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HYDRAULICS

BY

HORACE W. KING

Professor of Hydraulic Engineering,
University of Michigan

AND

CHESTER O. WISLER

Associate Professor of Hydraulic Engineering,
University of Michigan

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PREFACE

THIS book deals with the fundamental principles of hydraulics and their application in engineering practice. Though many formulas applicable to different types of problems are given, it has been the aim of the authors to bring out clearly and logically the underlying principles which form the basis of such formulas rather than to emphasize the importance of the formulas themselves.

Our present knowledge of fluid friction has been derived largely through experimental investigation and this has resulted in the development of a large number of empirical formulas. Many of these formulas have necessarily been included but, in so far as possible, the base formulas to which empirical coefficients have been applied have been derived analytically from fundamental consideration of basic principles.

The book is designed as a text for beginning courses in hydraulics and as a reference book for engineers who may be interested in the fundamental principles of the subject. Tables of coefficients are given which are sufficiently complete for classroom work, but the engineer in practice will need to supplement them with the results of his own experience and with data obtained from other published sources.

Chapters I to VI inclusive and Chapter XI were written by Professor Wisler and Chapters VII to X inclusive were written by Professor King. Acknowldgement for material taken from many publications is made at the proper place in the text.

H. W. K.
C. O. W.

University of Michigan,
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HYDRAULICS

CHAPTER I

INTRODUCTION

1. Fluids.—Fluids are substances which possess unlimited mobility and which offer practically no resistance to change of form. A perfect fluid yields to the slightest tangential stress, and can therefore have no tangential stress if it is at rest. Fluids may be divided into two classes, (*a*) liquids, or fluids that are practically incompressible, and (*b*) gases, or fluids that are highly compressible.

2. Definitions.—*Hydraulics* is the science embodying the laws that relate to the behavior of liquids, and particularly of water. In its original sense the term hydraulics was applied only to the flow of water in conduits, but the scope of the word has been broadened by usage.

Hydraulics may be divided conveniently into three branches: (*a*) *hydrostatics*, which deals with liquids at rest, (*b*) *hydrokinetics*, which treats of the laws governing the flow of liquids, and (*c*) *hydrodynamics*, which relates to the forces exerted upon other objects by liquids in motion or upon liquids by other objects in motion.

The fundamental laws of hydraulics apply equally to all liquids, but in hydrokinetics empirical coefficients must be modified to conform to the liquid considered. Water is the most common liquid and the only one that is of general interest to engineers.

3. Units Used in Hydraulics.—It is common practice in the United States and Great Britain to base hydraulic computations on the foot-pound-second system of units. In practically all hydraulic formulas these units are used, and if not otherwise

stated they are understood. Frequently the diameters of pipes or orifices are expressed in inches, pressures are usually stated in pounds per square inch, and volumes may be expressed in gallons. Before applying such data to problems, conversion to the foot-pound-second system of units should be made. Care in the conversion of units is essential. Errors in hydraulic computations result more frequently from wrong use of units than from any other cause.

4. Weight of Water.—Water has its maximum density at a temperature of 39.3° F. At this temperature pure water has been given a specific gravity of unity and it thus serves as a standard of density for all substances. The density of water decreases for temperatures above and below 39.3°. It freezes at 32° and boils at 212° F. The weight of pure water at its temperature of maximum density is 62.424 lbs. per cubic foot. The weights at various temperatures are given in the following table:

WEIGHT OF PURE WATER

Temperature, Fahrenheit	Pounds per cubic foot	Temperature, Fahrenheit	Pounds per cubic foot	Temperature, Fahrenheit	Pounds per cubic foot
32°	62.416	90°	62.118	160°	61.006
39.3	62.424	100	61.998	170	60.799
40	62.423	110	61.865	180	60.586
50	62.408	120	61.719	190	60.365
60	62.366	130	61.555	200	60.135
70	62.300	140	61.386	210	59.893
80	62.217	150	61.203	212	59.843

For temperatures above the boiling point Rankine gives the following approximate formula: w being the weight of water in pounds per cubic foot and T the temperature in degrees Fahrenheit,

$$w=\frac{124.85}{\dfrac{T+461}{500}+\dfrac{500}{T+461}} \quad \ldots\ldots\ldots \quad (1)$$

As water occurs in nature, it invariably contains a certain amount of salts and mineral matter in solution. Silt or other

impurities may also be carried in suspension. These substances are invariably heavier than water and they therefore increase its weight. The impurities contained in rivers, inland lakes and ordinary ground waters do not usually add more than one-tenth of a pound to the weight per cubic foot. Ocean water weighs about 64 lbs. per cubic foot. After long-continued droughts the waters of Great Salt Lake and of the Dead Sea have been found to weigh as much as 75 lbs. per cubic foot.

Since the weight of inland water is not greatly affected by ordinary impurities nor changes of temperature, an average weight of water may be used which usually will be close enough for hydraulic computations. In this book the weight of a cubic foot of water is taken as 62.4 lbs. Sea water will be assumed to weigh 64.0 lbs. per cubic foot unless otherwise specified. In very precise work weights corresponding to different temperatures may be taken from the above table.

5. Compressibility of Water.—Water is commonly assumed to be incompressible, but in reality it is slightly compressible. Upon release from pressure water immediately regains its original volume. For ordinary pressures the modulus of elasticity is constant—that is, the amount of compression is directly proportional to the pressure applied. The modulus of elasticity, E, varies with the temperature as shown in the following table.

Temperature, Fahrenheit	Modulus of elasticity, pounds per square inch
35°	288,000
77	327,000
212	360,000

These values hold only for pressures below 1000 lbs. per square inch. Hite obtained a reduction in volume of 10 per cent for a pressure of 65,000 lbs. per square inch, giving a value of E of 650,000 for this high intensity of pressure.

The compressibility of water usually affects the solution of practical problems in hydraulics only by changing its unit weight. Since pressures commonly encountered are relatively small, in most cases water may be considered incompressible without introducing any appreciable error.

6. Viscosity.—One of the characteristic properties of a liquid is its ability to flow. A perfect liquid would be one in which every particle could move in contact with adjacent particles without friction. The pressures between all such particles would be normal to their respective surfaces at the points of contact since there could be no tangential stress without friction. All liquids are capable, however, of resisting a certain amount of tangential stress, and the extent to which they possess this ability is a measure of their viscosity.

Water is one of the least viscous of all liquids. Oil, molasses and wax are examples of liquids having greater viscosity.

7. Surface Tension.—At any point within a body of liquid the molecules are attracted towards each other by equal forces. The molecules forming the surface layer, however, are subjected to an attraction downward that is not balanced by an upward attraction. This causes a film or skin to form on the surface and results in many interesting phenomena. A needle may be made to float upon water so long as the surface film is not broken, but it will sink immediately when the film is broken. Surface tension causes the spherical shape of dewdrops or drops of rain. This phenomenon also makes possible the hook gage described in Art. 86. Where water flows in an open conduit, surface tension retards velocities at the surface, the maximum velocity ordinarily being below the surface (see Art. 110). Capillary action is also explained by the phenomenon of surface tension combined with that of adhesion.

A (Fig. 1) illustrates an open tube of small diameter immersed in a liquid that wets the tube. Water rises in the tube higher than the level outside, the meniscus being concave upward. The tube *B* is immersed in mercury or some other liquid which does not wet the tube. In this case the meniscus is convex upward and the level of the liquid in the tube is depressed. The effect of capillarity decreases as the size of tube increases. The water in a tube one-half inch in diameter is approximately at the same level as the outside water, but it is appreciably different for smaller tubes. For this reason,

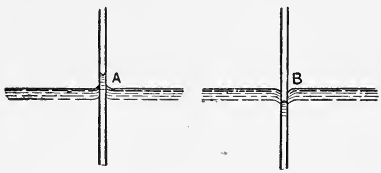

Fig. 1.

piezometer tubes (Art. 21) should not have diameters much smaller than one-half inch.

8. Accuracy of Computations.—Accuracy of computations is most desirable, but results should not be carried out to a greater number of significant figures than the data justify. Doing this implies an accuracy which does not exist and may give results that are entirely misleading.

Suppose, for example, that it be desired to determine the theoretical horse-power available in a stream where the discharge is 311 cu. ft. per second and the available head is 12.0 ft. The formula to be used is

$$\text{H.P.}=\frac{wQH}{550},$$

where w = the weight of a cubic foot of water;
Q = the discharge in cubic feet per second;
H = the available head.

Substituting in the formula,

$$\frac{62.4\times 311\times 12.0}{550}=423.41 \text{ horse-power.}$$

Referring to the table, page 2, it is seen that for the ordinary range of temperatures the weight of water may vary from 62.30 to 62.42 lbs. per cubic foot. Furthermore, the statement that the discharge is 311 cu. ft. per second means merely that the exact value is more nearly 311 than 310 or 312. In other words, the true value may lie anywhere between 310.5 and 311.5. Likewise, the fact that the head is given as 12.0 merely indicates that the correct value lies somewhere between 11.95 and 12.05. Therefore substituting in the formula the lower of these values,

$$\frac{62.30\times 310.5\times 11.95}{550}=420.30 \text{ horse-power.}$$

Again substituting in the formula the higher of the possible values,

$$\frac{62.42\times 311.5\times 12.05}{550}=426.00 \text{ horse-power.}$$

It is evident, therefore, that the decimal .41 in the original answer 423.41 is unjustified, and that the last whole number, 3,

merely represents the most probable value, since the correct value may lie anywhere between 420.30 and 426.00. The answer should, therefore, be given as 423.

It may be stated in general that in any computation involving multiplication or division, in which one or more of the numbers is the result of observation, the answer should contain the same number of significant figures as is contained in the observed quantity having the fewest significant figures. In applying this rule it should be understood that the last significant figure in the answer is not necessarily correct, but represents merely the most probable value. To give in the answer a greater number of significant figures indicates a degree of accuracy that is unwarranted and misleading.

CHAPTER II

PRINCIPLES OF HYDROSTATIC PRESSURE

9. Intensity of Pressure.—The intensity of pressure at any point in a liquid is the amount of pressure per unit area.

If the intensity of pressure is the same at every point on any area, A,

$$p=\frac{P}{A}, \quad = \frac{F}{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

the symbol p representing the intensity of pressure and P the total pressure acting upon the area.

If, however, the intensity of pressure is different at different points, the intensity of pressure at any point will be equal to the pressure on a small differential area surrounding the point divided by the differential area, or

$$p=\frac{dP}{dA}. = \frac{F}{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Intensities of pressure are commonly expressed in pounds per square inch and pounds per square foot. Where there is no danger of ambiguity, the term pressure is often used as an abbreviated expression for intensity of pressure.

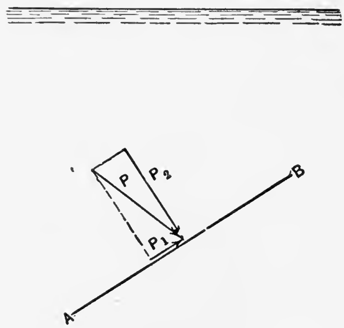

Fig. 2.

10. Direction of Resultant Pressure.—The resultant pressure on any plane in a liquid at rest is normal to that plane.

Assume that the resultant pressure P, on any plane AB (Fig. 2), makes an angle other than 90° with the plane. Resolving P into rectangular components P_1 and P_2, respectively parallel

with and perpendicular to AB, gives a component P_1 which can be resisted only by a tangential stress. By definition, a liquid at rest can resist no tangential stress and therefore the pressure must be normal to the plane. This means that there can be no static friction in hydraulics.

11. Pascal's Law.—At any point within a liquid at rest, the intensity of pressure is the same in all directions. This principle is known as *Pascal's Law.*

Consider an infinitesimally small wedge-shaped volume, BCD (Fig. 3), in which the side BC is vertical, CD is horizontal and BD makes any angle θ with the horizontal. Let A_1, A_2 and A_3 and p_1, p_2 and p_3 represent, respectively the areas of these sides and the intensities of pressure to which they are subjected. Assume that the ends of the wedge are vertical and parallel.

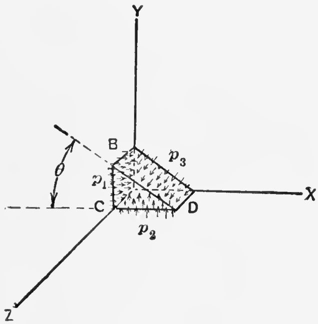

Fig. 3.

Since the wedge is at the rest, the principles of equilibrium may be applied to it. From Art. 10 it is known that the pressures are normal to the faces of the wedge. Choosing the coordinate axes as indicated in Fig. 3 and setting up the equations of equilibrium, $\Sigma X=0$ and $\Sigma Y=0$, and neglecting the pressures on the ends of the wedge, since they are the only forces acting on the wedge which have components along the Z-axis and therefore balance each other, the following expressions result;

$$p_1A_1=p_3A_3\sin\theta,$$

$$p_2A_2=p_3A_3\cos\theta.$$

But

$$A_3\sin\theta=A_1 \quad \text{and} \quad A_3\cos\theta=A_2.$$

Therefore

$$p_1=p_2=p_3.$$

Since BD represents a plane making any angle with the horizontal and the wedge is infinitesimally small so that the sides may be considered as bounding a point, it is evident that the intensity of pressure at any point must be the same in all directions.

12. Free Surface of a Liquid.—Strictly speaking, a liquid having a *free surface* is one on whose surface there is absolutely no pressure. It will be shown later, however, that there is always some pressure on the surface of every liquid.

In practice the free surface of a liquid is considered to be a surface that is not in contact with the cover of the containing vessel. Such a surface may or may not be subjected to the pressure of the atmosphere.

It may be shown that the free surface of a liquid at rest is horizontal. Assume a liquid having a surface which is not horizontal, such as $ABCDE$ (Fig. 4). A plane MN, inclined to the horizontal, may be passed through any liquid having such a surface in such manner that a portion of the liquid BCD lies above the plane. Since the liquid is at rest, BCD must be in equilibrium, but the vertical force of gravity would necessarily have a component along the inclined plane which could be resisted only by a tangential stress. As liquids are incapable of resisting tangential stress it follows that the free surface must be horizontal.

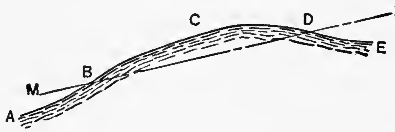

Fig. 4.

13. Atmospheric Pressure.—All gases possess mass and consequently have weight. The atmosphere, being a fluid composed of a mixture of gases, exerts a pressure on every surface with which it comes in contact. At sea level under normal conditions atmospheric pressure amounts to 2116 lbs. per square foot or about 14.7 lbs. per square inch.

Variation in Atmospheric Pressure with Altitude

Altitude above sea level in feet	Pressure in pounds per square inch	Altitude above sea level in feet	Pressure in pounds per square inch
0	14.7	5,280	12.0
1000	14.15	6,000	11.7
2000	13.6	7,000	11.3
3000	13.1	8,000	10.9
4000	12.6	9,000	10.5
5000	12.1	10,000	10.1

The intensity of atmospheric pressure decreases with the altitude. Owing to compressibility, the density of air also decreases with altitude, and therefore the intensity of pressure changes less rapidly as the altitude increases. The accompanying table gives approximate values of the atmospheric pressure corresponding to different elevations above sea level.

14. Vacuum.—A perfect vacuum, that is, a space in which there is no matter either in the solid, liquid or gaseous form, has never been obtained. It is not difficult, however, to obtain a space containing a minute quantity of matter. A space in contact with a liquid, if it contains no other substance, always contains vapor from that liquid. In a perfect vacuum there could be no pressure.

In practice, the word "vacuum" is used frequently in connection with any space having a pressure less than atmospheric pressure, and the term "amount of vacuum" means the amount the pressure is less than atmospheric pressure. The amount of vacuum is usually expressed in inches of mercury column or in pounds per square inch measured from atmospheric pressure as a base. For example, if the pressure within a vessel is reduced to 12 lbs. per square inch, which is equivalent to 24.5 in. of mercury column, there is said to be a vacuum of 2.7 lbs. per square inch or 5.5 inches of mercury. (Arts. 20 and 22.)

15. Absolute and Gage Pressure.—The intensity of pressure above absolute zero is called *absolute* pressure. Obviously, a negative absolute pressure is impossible.

Usually pressure gages are designed to measure the intensities of pressure above or below atmospheric pressure as a base. Pressures so measured are called *relative* or *gage* pressures. Negative gage pressures indicate the amount of vacuum, and at sea level pressures as low as, but no lower than, -14.7 lbs. per square inch are possible. Absolute pressure is always equal to gage pressure plus atmospheric pressure.

Fig. 5 illustrates a gage dial, on the inner circle of which is shown the ordinary gage and vacuum scale. The outer scale indicates the corresponding absolute pressures.

16. Intensity of Pressure at any Point.—To determine the intensity of pressure at any point in a liquid at rest or the variation in pressure in such a liquid, consider any two points such as 1 and 2 (Fig. 6) whose vertical depths below the free surface of the

liquid are h_1 and h_2, respectively. Consider that these points lie in the ends of an elementary prism of the liquid, having a cross-sectional area dA and length l. Since this prism is at rest, all of the forces acting upon it must be in equilibrium. These forces consist of the fluid pressure on the sides and ends of the prism and the force of gravity.

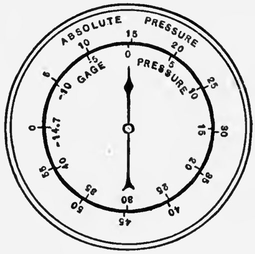

FIG. 5.—Gage dial.

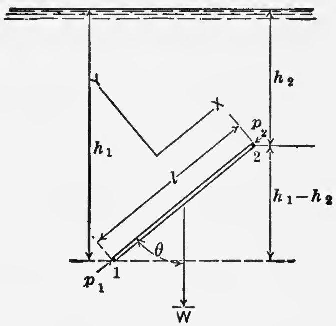

FIG. 6.

Let X and Y, the coordinate axes, be respectively parallel with and perpendicular to the axis of the prism which makes an angle θ with the vertical. Also let p_1 and p_2 be the intensities of pressure at points 1 and 2, respectively, and w be the unit weight of the liquid.

Considering forces acting to the left along the X-axis as negative and remembering that the pressures on the sides of the prism are normal to the X-axis and therefore have no X components, the following equation may be written:

$$\Sigma X = p_1 dA - p_2 dA - wl dA \cos \theta = 0.$$

Since $l \cos \theta = h_1 - h_2$, this reduces to

$$p_1 - p_2 = w(h_1 - h_2). \quad \quad (3)$$

From this equation it is evident that in any liquid the difference in pressure between any two points is the product of the unit weight of the liquid and the difference in elevation of the points.

If $h_1 = h_2$, p_1 must equal p_2; or, in other words, in any continuous homogeneous body of liquid at rest, the intensities of pressure at all points in a horizontal plane must be the same. Stated con-

versely, in any homogeneous liquid at rest all points having equal intensities of pressure must lie in a horizontal plane.

If in equation (3) h_2 is made equal to zero, p_2 is the intensity of pressure on the liquid surface. In case that pressure is atmospheric, or p_a, equation (3) becomes

$$p_1 = wh_1 + p_a, \qquad (4)$$

or, in general,

$$p = wh + p_a. \qquad (5)$$

In this equation p is evidently the absolute pressure at any point in the liquid at a depth h below the free surface. The corresponding gage pressure is

$$p = wh. \qquad (6)$$

In the use of the above equations care must be taken to express all of the factors involved in their proper units. Unless otherwise stated p will always be understood to be intensity of pressure in pounds per square foot, w will be the weight of a cubic foot of the liquid and h will be measured in feet.

At any point in a body of water at a depth h below the free surface, the absolute pressure in pounds per square foot is

$$p = 62.4h + 2116. \qquad (7)$$

The relative or gage pressure in pounds per square foot is

$$p = 62.4h. \qquad (8)$$

If, however, it is desired to express the pressure in pounds per square inch it is necessary only to divide through by 144. Hence if p' is used to express absolute pressure in pounds per square inch,

$$p' = \frac{p}{144} = \frac{62.4}{144}h + \frac{2116}{144}$$

$$= .433h + 14.7,$$

or, expressed as gage pressure in pounds per square inch,

$$p' = .433h.$$

17. Pressure Head.—Equation (6) may be written in the form,

$$\frac{p}{w}=h. \qquad (9)$$

Here h, or its equivalent, $\frac{p}{w}$, represents the height of a column of liquid of unit weight w that will produce an intensity of pressure, p. It is therefore called *pressure head.*

In considering water pressures the pressure head, h, is expressed in feet of water column regardless of whether it is obtained by dividing the pressure in pounds per square foot by 62.4 or by dividing the pressure in pounds per square inch by 0.433.

18. Transmission of Pressure.—Writing equation (3) in the form,

$$p_1=p_2+w(h_1-h_2), \qquad (10)$$

it is evident that the pressure at any point, such as point 1 (Fig. 6), in a liquid at rest is equal to the pressure at any other point, such as point 2, plus the pressure produced by a column of the liquid whose height, h, is equal to the difference in elevation between the two points. Any change in the intensity of pressure at point 2 would cause an equal change at point 1. In other words, a pressure applied at any point in a liquid at rest is transmitted equally and undiminished to every other point in the liquid.

This principle is made use of in the hydraulic jack by means of which heavy weights are lifted by the application of relatively small forces.

Example.—In Fig. 7 assume that the piston and weight, W, are at the same elevation, the face of the piston having an area of 2 sq. in. and the face of the weight 20 sq. in. What weight W can be lifted by a force P of 100 lbs. applied at the end of the lever as shown in the figure?

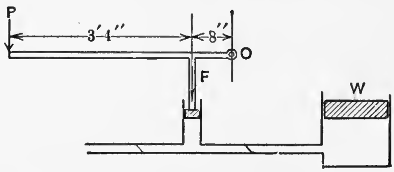

Fig. 7.—Hydraulic jack.

Since atmospheric pressure is acting on both the piston and weight its resultant effect will be zero and it may therefore be

neglected. Taking moments about O the force F on the piston is

$$\tfrac{2}{3}F = 4 \times 100$$

$$F = 600 \text{ lbs.}$$

$$\frac{600}{2} = 300 \text{ lbs. per square inch.}$$

which is the intensity of pressure on the face of the piston, and since the two are at the same elevation in a homogeneous liquid at rest it is also the intensity of pressure on the weight. Therefore

$$W = 20 \times 300$$

$$= 6000 \text{ lbs.}$$

Evidently this is the value of W for equilibrium; any weight less than 6000 pounds could be lifted by the force of 100 lbs.

19. Vapor Pressure.—Whenever the free surface of any liquid is exposed to the atmosphere, evaporation is continually taking place. If, however, the surface is in contact with an enclosed space, evaporation takes place only until the space becomes saturated with vapor. This vapor produces a pressure, the amount of which depends only upon the temperature and is entirely independent of the presence or absence of air or other gas within the enclosed space. The pressure exerted by a vapor within a closed space is called vapor pressure.

In Fig. 8, A represents a tube having its open end submerged in water and a stopcock at its upper end. Consider the air within A to be absolutely dry at the time the stopcock is closed. At the instant of closure the water surfaces inside and outside the tube will stand at the same level. Evaporation within the tube, however, will soon saturate the space containing air and create a vapor pressure, p_v, which will cause a depression of the water surface within the vessel equal to $\frac{p_v}{w}$.

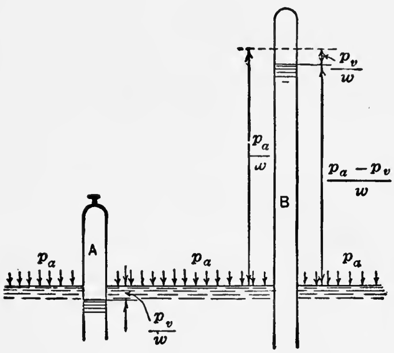

FIG. 8.

In the same figure, B represents a tube closed at the upper end. Assume a perfect vacuum in the space above the water in the tube. If this condition were possible the water level in B would stand at an elevation $\frac{p_a}{w}$ above the water surface outside. Vapor pressure within the vessel, however, causes a depression $\frac{p_v}{w}$ exactly equal to that produced within A, so that the maximum height of water column possible under conditions of equilibrium in such a tube is $\frac{p_a - p_v}{w}$. Vapor pressures increase with the temperature, as is shown in the following table.

VAPOR PRESSURES OF WATER IN FEET OF WATER COLUMN

Temperature, F.	$\frac{p_v}{w}$	Temperature, F.	$\frac{p_v}{w}$	Temperature, F.	$\frac{p_v}{w}$
−20°	0.02	60°	0.59	140°	6.63
−10	.03	70	0.83	150	8.54
0	.05	80	1.16	160	10.90
10	.08	90	1.59	170	13.78
20	.13	100	2.17	180	17.28
30	.19	110	2.91	190	21.49
40	.28	120	3.87	200	26.52
50	.41	130	5.09	212	33.84

20. The Mercury Barometer.—The barometer is a device for measuring intensities of pressure exerted by the atmosphere. In 1643 Torricelli discovered that if a tube (Fig. 9) over 30 in. long and closed at one end, is filled with mercury and then made to stand vertically with the open end submerged in a vessel of mercury, the column in the tube will stand approximately 30 in. above the surface of the mercury in the vessel. Such a device is known as a mercury barometer. Pascal proved that the height of the column of mercury depended upon the atmospheric pressure, when he carried a barometer to a higher elevation and found that the height of the column decreased as the altitude increased.

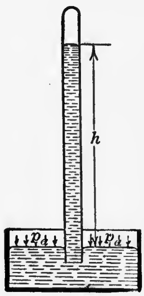

FIG. 9.—Mercury barometer.

Although theoretically water or any other liquid may be used for barometers, two difficulties arise in using water. First, the height of water column necessary to balance an atmospheric pressure of 14.7 lbs. per square inch is about 34 feet at sea level, which height is too great for convenient use; and, second, as shown in Art. 19, water vapor collecting in the upper portion of the tube creates a pressure which partially balances the atmospheric pressure, so that the barometer does not indicate the total atmospheric pressure.

Since mercury is the heaviest known liquid and has a very low vapor pressure at ordinary air temperatures it is more satisfactory for use in barometers than any other liquid.

21. Piezometer Tubes.—A piezometer tube is a tube tapped into the wall of a vessel or pipe for the purpose of measuring moderate pressures. Thus the height of water column in tube *a* (Fig. 10) is a measure of the pressure at *A*, the top of the pipe. Similarly the pressure at the elevation *B* is measured by the height of water column in tube *b*, that is, $p = wh$. Piezometer tubes always measure gage pressures since the water surface in the tube is subjected to atmospheric pressure. Obviously, the level to which water will rise in a tube will be the same regardless of whether the connection is made in the side, bottom or cover of the containing vessel.

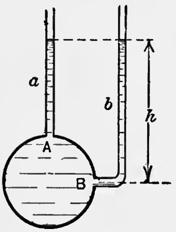

Fig. 10.—Cross-section of pipe with piezometer tubes.

Piezometer tubes are also used to measure pressure heads in pipes where the water is in motion. Such tubes should enter the pipe in a direction at right angles to the direction of flow and the connecting end should be flush with the inner surface of the pipe. If these precautions are not observed, the height of water column may be affected by the velocity of the water, the action being similar to that which occurs in Pitot tubes. (See Art. 48.)

In order to avoid the effects of capillary action, piezometer tubes should be at least $\frac{1}{2}$ in. in diameter.

Pressures less than atmospheric pressure may be measured by either of the methods illustrated in Fig. 11 which shows a pipe section AB in which the water is flowing. The vertical distance, *h*, which the water surface, *C*, in the open tube drops below *A* is a measure of the pressure below atmospheric pressure, or, in

other words, it is a measure of the amount of vacuum existing at A. This is true since the pressure at C' is atmospheric, being at the same level as C in a homogeneous liquid at rest and the pressure at A must be less than at C' by the amount wh. (See Art. 17.)

To the right in Fig. 11 is shown an inverted piezometer tube with the lower end immersed in an open vessel containing water. Atmospheric pressure acting on the water surface in the vessel forces water to rise in the tube to a height h which measures the vacuum existing at B. Air will be drawn into the pipe at B until the intensity of pressure on the water surface in the tube equals the pressure at B. Neglecting the weight of the air in the tube, it is evident that $p_B = p_D = p_a - wh$. Here p_B is expressed as absolute pressure since atmospheric pressure is included in the equation.

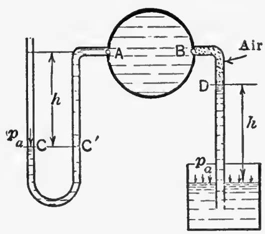

FIG. 11.

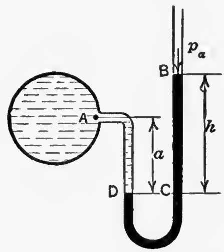

FIG. 12.—Mercury gage.

22. Mercury Gage.—In the measurement of pressures so great that the length of tube required for a water piezometer would be unwieldy, the mercury U-tube, illustrated in Fig. 12, is a convenient substitute.

Water under pressure fills the pipe, or vessel at A and the tube down to the level D. Mercury fills the tube from D around to B, above which level the tube is open to the atmosphere.

The pressure at C equals the pressure at B which is atmospheric, or p_a, plus the pressure produced by the mercury column, h. Hence,

$$p_C = p_a + w'h. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

If p_C and p_a are expressed in pounds per square foot and h in feet, w' is the weight of a cubic foot of mercury, or 13.6 (specific

gravity of mercury)$\times 62.4=848$ lbs. If, however, p_C and p_a are expressed in pounds per square inch and h in feet,

$$w'=\frac{848}{144}\ (\text{or } 13.6\times 0.433)=5.89.$$

The pressure at D (Fig. 12) is the same as at C, being at the same level in a homogeneous liquid at rest. The pressure at A is equal to that at C minus the pressure produced by the water column a, or,

$$p_A=p_C-wa. \qquad (12)$$

Here again if p_A and p_C are expressed in pounds per square foot and a in feet, w equals 62.4, but if p_A and p_C are expressed in pounds per square inch and a in feet, w equals 0.433. Combining equations (11) and (12),

$$p_A=p_a+w'h-wa. \qquad (13)$$

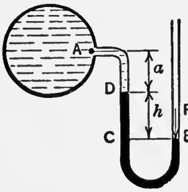

FIG. 13.

Here p_A is expressed as absolute pressure, since p_a (atmospheric pressure) enters into the equation.

If p_A is less than atmospheric pressure by an amount greater than wa, h is negative, as in Fig. 13, and

$$p_A=p_a-w'h-wa. \qquad (14)$$

23. The Differential Gage.—The differential gage as the name indicates, is used only for measuring *differences* in pressure. A liquid heavier or lighter than water is used in the gage, depending upon whether the differences in pressure to be measured are great or small.

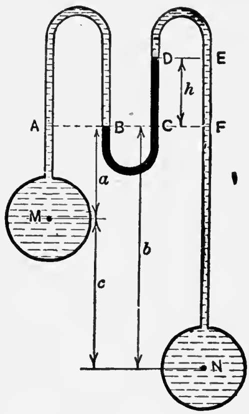

FIG. 14.—Differential mercury gage.

In Fig. 14 is shown the form of differential gage usually employed for measuring large differences in pressure. M and N are two pipes containing water under different pressures which may be either greater or less than atmospheric pressure. The two pipes are connected by a bent tube, of which the portion BCD is filled with mercury, while all of the remaining space is filled with water.

If M and N were at the same elevation the difference in pressure in the two pipes would be measured by the pressure due to the mercury column DC minus that due to the water column EF or would equal $w'h-wh$.

If the two pipes are at different elevations, to the above difference in pressure must be added or subtracted the intensity of pressure produced by the water column whose height is equal to the difference in elevation of the pipes. Proof of these statements follows.

The pressure at A equals that at M minus the pressure produced by the water column whose height is a. Evidently the pressure at B is the same as at A, being at the same elevation in a homogeneous liquid at rest. For the same reason the pressures at B and C are also equal. Hence,

$$p_C=p_B=p_A=p_M-wa. \qquad (15)$$

The pressures at C and F are not equal, since these points are not connected by a homogeneous liquid.

The pressure at D is equal to that at C minus the pressure produced by the mercury column h and is

$$p_D=p_C-w'h. \qquad (16)$$

$$p_E=p_D$$

and

$$p_N=p_E+w(h+b).$$

Combining these equations

$$p_M-wa-w'h+wh+wb=p_N, \qquad (17)$$

or

$$p_M-p_N=w'h-wh-w(b-a). \qquad (18)$$

Obviously, the greater the difference between w' and w the greater the difference in pressure that can be measured for any given value of h.

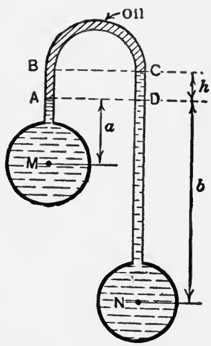

Fig. 15.—Differential oil gage.

Fig. 15 illustrates a type of differential gage used when the difference in pressures to be measured is small. Usually a liquid, such as a light oil, whose specific gravity is slightly less than unity is used in the upper portion of the inverted U-tube, AC, the remainder of the tube being filled with water.

As with the mercury gage, if M and N are at the same elevation the difference in pressure will be equal to that produced by the oil column AB and the water column CD. If M and N are at different levels, then

$$p_M - wa - w'h + wh + wb = p_N \quad . \quad . \quad . \quad . \quad (19)$$

or

$$p_M - p_N = w'h - wh - w(b-a), \quad . \quad . \quad . \quad . \quad (20)$$

the equations being the same as were obtained for the mercury differential gage.

The use of a liquid whose unit weight w' is very nearly the same as that of water makes a very sensitive measuring device. With such a device small differences in pressure will produce relatively large values of h.

The value of h produced by any particular difference in pressure is independent of the relative cross-sectional areas of the columns AB and CD. This is evident since it is the difference in *intensities* of pressure that is measured and not difference in *total* pressures.

24. Suction Pumps and Siphons.—Suction pumps depend upon atmospheric pressure for their operation. The plunger creates a partial vacuum in the pump stock, and atmospheric pressure acting upon the outer water surface causes water to rise within the pump.

The operation of siphons is also produced by atmospheric pressure. In Fig. 16 the two vessels, A and B, are connected by a tube. As long as the tube is filled with air there is no tendency for water to flow. If, however, air is exhausted from the tube at C, atmospheric pressure will cause water to rise in each leg of the tube an equal height above the water surfaces. When water has been drawn up a distance d, equal to the height of the summit above the water surface in the upper vessel flow from A to B will begin. If the velocity of the water is high enough, any air entrapped in the tube will be carried out by the moving water, and the tube will flow full. If the sum-

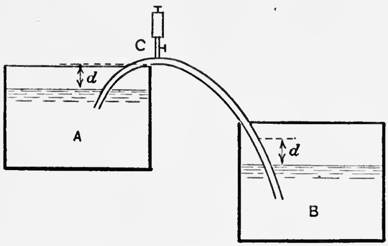

FIG. 16.—Siphon.

mit of the siphon is a distance greater than $\frac{p_a - p_v}{w}$ above the water surface in the higher vessel, siphon action is impossible.

It is not necessary that the discharge end of the tube be submerged to induce siphon action. If flow is started by suction at the free end of the tube or by other means, it will continue as long as the discharge end of the tube is lower than the water surface in the vessel or until the vacuum in the siphon is broken.

PROBLEMS

1. Determine the intensity of pressure on the face of a dam at a point 40 ft. below the water surface. 46ft H_2O or $\frac{40}{2.3}$ = #/□"

(*a*) Expressed in pounds per square foot gage pressure.
(*b*) Expressed in pounds per square inch gage pressure.
(*c*) Expressed in pounds per square foot absolute pressure.
(*d*) Expressed in pounds per square inch absolute pressure.

2. Determine the intensity of pressure in a vessel of mercury (sp. gr. = 13.6) at a point 8 in. below the surface, expressing the answer in the same units as in Problem 1.

3. A vertical pipe, 100 ft. long and 1 in. in diameter, has its lower end open and flush with the inner surface of the cover of a box, 2 ft. square and 6 in. high. The bottom of the box is horizontal. Neglecting the weight of the pipe and box, both of which are filled with water, determine:

(*a*) The total hydrostatic pressure on the bottom of the box.
(*b*) The total pressure exerted on the floor on which the box rests.

4. At what height will water stand in a water barometer at an altitude of 5000 ft. above sea level if the temperature of the water is 70° F.? Under similar conditions what would be the reading of a mercury barometer, neglecting the vapor pressure of mercury?

5. What are the absolute and gage pressures in pounds per square inch existing in the upper end of the water barometer under the conditions of Problem 4?

6. What height of mercury column will cause an intensity of pressure of 100 lbs. per square inch? What is the equivalent height of water column?

7. A pipe 1 in. in diameter is connected with a cylinder 24 in. in diameter, each being horizontal and fitted with pistons. The space between the pistons is filled with water. Neglecting friction, what force will have to be applied to the larger piston to balance a force of 20 lbs. applied to the smaller piston?

8. In Problem 7, one leg of a mercury U-tube is connected with the smaller cylinder. The mercury in this leg stands 30 in. below the center of the pipe, the intervening space being filled with water. What is the height of mercury in the other leg, the end of which is open to the air?

9. A U-tube with both ends open to the atmosphere contains mercury in the lower portion. In one leg, water stands 30 in. above the surface of the mercury; in the other leg, oil (sp. gr. = 0.80) stands 18 in. above the surface

of the mercury. What is the difference in elevation between the surf the oil and water columns?

10. Referring to Fig. 15, page 19, if the pressure at M is 20 lbs. per inch, what is the corresponding pressure at N if $a=1$ ft., $b=4$ ft. and h (Sp. gr. of oil $=0.80$.)

11. In Fig. 15, page 19, determine the value of h, if $a=1$ ft., $b=4$ ft. a pressure at N is 1.4 lbs. per square inch greater than at M. (Sp. gr 0.80.)

12. A vertical tube 10 ft. long, with its upper end closed and low open, has its lower end submerged 4 ft. in a tank of water. Neglecting pressure, how much will the water level in the tube be below the level tank?

13. In Fig. 7, page 13, if the diameters of the two cylinders are 3 24 in. and the face of the smaller piston is 20 ft. above the face of the W, what force P is required to maintain equilibrium if $W=8000$ lbs.?

14. Referring to Fig. 12, page 17, if $h=20$ in. and $a=12$ in., what absolute pressure in pounds per square inch at A? What is the gage pr

15. In Problem 14, if the surface of the mercury column in each leg U-tube stands at the same elevation as A when the pressure at A is pheric, determine the values of a and h when the gage pressure at A is per square inch, the diameter of the tube being the same throughout.

16. Referring to Fig. 13, page 18, determine the absolute press pounds per square inch at A when $a=8$ in. and $h=10$ in. What is the sponding gage pressure?

17. In Fig. 14, page 18, let $c=6$ ft., and assume that $h=0$ wh pressure at M is atmospheric. If the pressure at N remains constant, det the value of h when the gage pressure at M is increased to 8 lbs. per inch.

18 In Fig. 14, page 18, if $a=24$ in. and $c=6$ ft., what is the valu when the pressure at M is 10 lbs. per square inch greater than at N.

19. A and B are, respectively, the closed and open ends of a U both being at the same elevation. For a distance of 18 in. below tube is filled with oil (sp. gr. $=0.8$); for a distance of 3 ft. below B, th is filled with water, on the surface of which atmospheric pressure is The remainder of the tube is filled with mercury. What is the al pressure at A expressed in pounds per square inch?

20. In Problem 19, if B were closed and A were open to the atmos what would be the gage pressure at B, expressed in pounds per square

CHAPTER III

PRESSURE ON SURFACES

25. Total Pressure on Plane Areas.—The total pressure on any plane surface is equal to the product of its area and the intensity of pressure at its center of gravity. This may be proved as follows:

Fig. 17 shows projections on two vertical planes normal to

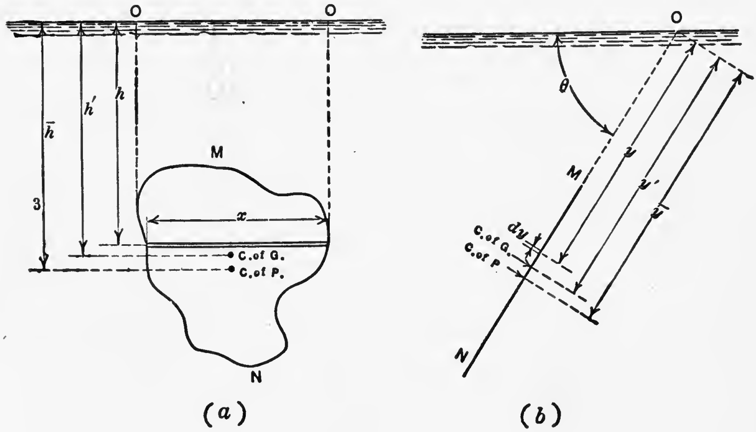

FIG. 17.

each other, of any plane surface, MN, subjected to the full static pressure of a liquid with a free surface. Projection (b) is on a plane at right angles to MN. The surface MN makes any angle, θ, with the horizontal, and, extended upward, the plane of this surface intersects the surface of the liquid in the line OO, shown as the point O in (b).

Consider the surface MN to be made up of an infinite number of horizontal strips each having a width dy so small that the

intensity of pressure on it may be considered constant. The area of any strip whose length is x, is

$$dA = xdy.$$

The liquid having a unit weight of w, the intensity of pressure on any strip at a depth h below the surface and at a distance y from the line OO is

$$p = wh = wy \sin \theta.$$

The total pressure on the strip is

$$dP = wy \sin \theta \, dA$$

and the total pressure on MN is

$$P = w \sin \theta \int y dA. \qquad (1)$$

From the definition of center of gravity,

$$\int y dA = Ay', \qquad (2)$$

where y' is the distance from the line OO to the center of gravity of A. Hence,

$$P = w \sin \theta \, Ay'. \qquad (3)$$

Since the vertical depth of the center of gravity below the water surface is

$$h' = y' \sin \theta, \qquad (4)$$

it follows that

$$P = wh'A, \qquad (5)$$

where wh' represents the intensity of pressure at the center of gravity of A.

26. Center of Pressure on Plane Areas.—Any plane surface subjected to hydrostatic pressure is acted upon by an infinite number of parallel forces whose magnitudes vary with the depth, below the free surface, of the various infinitesimal areas on which the respective forces act. Since these forces are parallel they may be replaced by a single resultant force P. The point on the surface at which this resultant force acts is called the *center of pressure*. In other words, if the total hydrostatic pressure on any area were applied at the center of pressure the same effect would be

produced on the area as is produced by the variable pressures distributed over the area.

The position of the horizontal line containing the center of pressure of a plane surface subjected to hydrostatic pressure may be determined by taking moments of all the forces acting on the area about some horizontal axis in the plane of the surface. For the case described in the preceding article and illustrated in Fig 17, the line OO may be taken as the axis of moments for the surface MN. Designating by $\bar{y}$ the distance to the center of pressure from the axis of moments, it follows from the definition of center of pressure that,

$$P\bar{y}=\int ydP, \qquad (6)$$

or

$$\bar{y}=\frac{\int ydP}{P}. \qquad (7)$$

It was shown in Art. 25 that

$$dP=wy \sin \theta \, dA$$

and

$$P=w \sin \theta \, Ay'. \qquad (3)$$

Substituting these values, equation (7) becomes

$$\bar{y}=\frac{w \sin \theta \int y^2 dA}{w \sin \theta \, Ay'}=\frac{\int y^2 dA}{Ay'}, \qquad (8)$$

in which $\int y^2 dA$ is the moment of inertia, I_0, of MN with respect to the axis OO, and Ay' is the statical moment, S, of MN with respect to the same axis.
Therefore,

$$\bar{y}=\frac{I_0}{S}. \qquad (9)$$

Since the moment of inertia of an area about any axis equals the moment of inertia of the area about a parallel axis through its center of gravity plus the product of the area and the square of the distance between the two axes, equation (9) may be written,

$$\bar{y}=\frac{Ak^2+Ay'^2}{Ay'}, \qquad (10)$$

or $$\bar{y}=y'+\frac{k^2}{y'}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where k is the radius of gyration of the area.

The above discussion refers only to the determination of the position of the horizontal line which contains the center of pressure—that is, $\bar{y}$ gives only the distance from the horizontal axis of moments to the center of pressure. For any figure such that the locus of the midpoints of the horizontal strips is a straight line, as, for instance, a triangle or trapezoid with base horizontal, the center of pressure falls on that straight line. It is with such figures that the engineer is usually concerned. For other figures, the horizontal location of the center of pressure may be found in

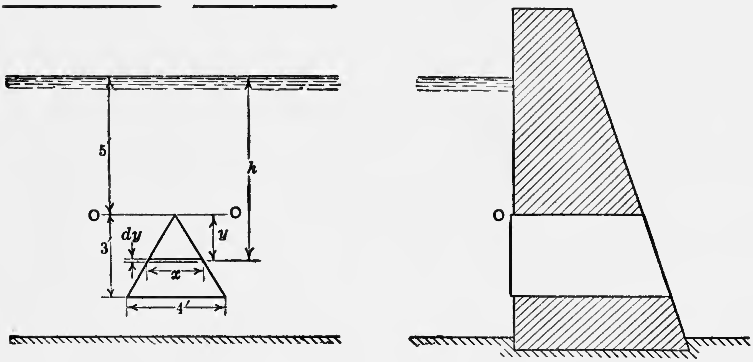

FIG. 18.

a manner similar to that described above by taking moments about an axis, within the plane of the surface, at right angles to the horizontal axis of moments.

Examples[1]: (*a*) Find the center of pressure on the vertical triangular gate shown in Fig. 18.

[1] It is apparent that the solution of any problem involving the location of the center of pressure, for an area whose radius of gyration is known or can be readily found, may be accomplished by the simple substitution of values for k and y' in equation (11). This involves a mere mathematical process with no necessity on the part of the student for either thought or understanding of fundamental principles. It is therefore recommended that the beginner solve all such problems by the use of equation (6) which is really nothing more than a formulated expression of the definition of center of pressure. The examples given are solved by this method.

Using the equation

$$P\bar{y}=\int y dP \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

it will be necessary to express dP in terms of y.

In this case it will be convenient to take moments about O-O, a horizontal line through the vertex of the gate, lying in the plane of the gate. Moments could, however, be taken about any other horizontal axis lying within this plane.

The total pressure dP on any thin horizontal strip at a distance y from the axis of moments equals the intensity of pressure, wh, times the area dA, or

$$dP=wh\,dA.$$

Since $w=62.4$, $h=5+y$ and $dA=x\,dy$

$$dP=62.4(5+y)x\,dy.$$

Since x varies with y it must be expressed in terms of y before integrating.

From similar triangles,

$$\frac{x}{4}=\frac{y}{3} \quad \text{or} \quad x=\frac{4}{3}y.$$

Substituting,

$$dP=62.4\times\tfrac{4}{3}\ (5y+y^2)dy$$

and

$$\int y dP=62.4\times\tfrac{4}{3}\int_0^3(5y^2+y^3)dy,$$

also

$$P=wh'A=62.4\times7\times6,$$

therefore

$$\bar{y}=\frac{62.4\times\frac{4}{3}\int_0^3(5y^2+y^3)dy}{62.4\times42},$$

$$=\tfrac{2}{63}\Big[\tfrac{5}{3}y^3+\tfrac{1}{4}y^4\Big]_0^3$$

$$=\tfrac{2}{63}(\tfrac{5}{3}\times27+\tfrac{1}{4}\times81)$$

$$=2\tfrac{1}{14}\text{ ft. below the vertex,}$$

or $\frac{1}{14}$ ft. below the center of gravity of the gate.

The horizontal location of the center of pressure in this case is on the median connecting the vertex with the base.

(*b*) Find the center of pressure on the inclined rectangular gate shown in Fig. 19.

Taking moments again about the top of the gate as an axis,

$$\int y\,dP=\int ywh\ dA,$$

where

$$h=5+y\cos 30°$$

$$=5+\frac{\sqrt{3}}{2}y,$$

and

$$dA=6dy$$

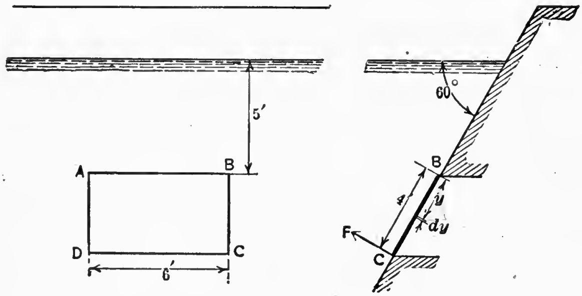

Fig. 19.

Since $P=wh'A=62.4(5+2\cos 30°)24=62.4\left(5+\frac{\sqrt{3}}{2}\times 2\right)24,$

$$\bar{y}=\frac{62.4\times 6\int_0^4\left(5y+\frac{\sqrt{3}}{2}y^2\right)dy}{62.4\times 24(5+\sqrt{3})}$$

$$=\frac{1}{4(5+\sqrt{3})}\left[\frac{5}{2}y^2+\frac{\sqrt{3}}{6}y^3\right]_0^4$$

$$=\frac{1}{4(5+\sqrt{3})}\left(\frac{5}{2}\times 16+\frac{\sqrt{3}}{6}\times 64\right)$$

$$=\frac{58.47}{26.93}$$

$$=2.17 \text{ ft. below } AB,$$

measured along the plane of the gate. The horizontal location of the center of pressure is 3 ft. from either end of the gate.

(*c*) In example (*b*) what force F applied normally to the gate at its lower edge will be required to open it?

Knowing the total pressure, P, on the gate and the location of the center of pressure, by taking moments about the upper edge which is the center of rotation,

$$4F = 2.17P$$

$$F = \frac{2.17}{4} \times 62.4 \times 24(5+\sqrt{3})$$

$$= 5470 \text{ lbs.}$$

The value of F can also be found directly without determining either the location of the center of pressure or the total pressure.

Taking moments about the top of the gate,

$$4F = \int_0^4 y\,dP$$

$$= \int_0^4 ywh\,dA$$

$$= 62.4 \times 6 \int_0^4 \left(5y + \frac{\sqrt{3}}{2}y^2\right) dy$$

$$= 62.4 \times 6 \left[\frac{5}{2}y^2 + \frac{\sqrt{3}}{6}y^3\right]_0^4$$

$$= 62.4 \times 6 \left(\frac{5}{2} \times 16 + \frac{\sqrt{3}}{6} \times 64\right)$$

$$= 21890$$

$$F = 5470 \text{ lbs.}$$

If this force were applied at the bottom of the gate, the gate would be in equilibrium and there would be no reaction on the supports along the lower edge or sides of the gate. Any force greater than 5470 lbs. would open the gate.

27. Graphical Method of Location of Center of Pressure.—Semigraphic methods may be used advantageously in locating the center of pressure on any plane area whose width is constant.

The rectangular surface, $ABCD$, illustrated in Fig. 19 is shown in perspective in Fig. 20a. BC (Fig. 20b) represents the projection of the rectangle on a vertical plane perpendicular to the plane of the surface. The vertical depths below the water surface of the top and bottom of the rectangle are, respectively, h_1 and h_2. The intensity of pressure, wh_1, on the top of the rectangle is represented by the equal ordinates AA' and BB' (Fig. 20a), and on the bottom of the rectangle the equal ordinates CC' and DD' represent the intensity of pressure wh_2.

BB' and CC' (Fig. 20b) represent, to a reduced scale, the intensities of pressure at B and C. BB'' and CC'' are laid off equal

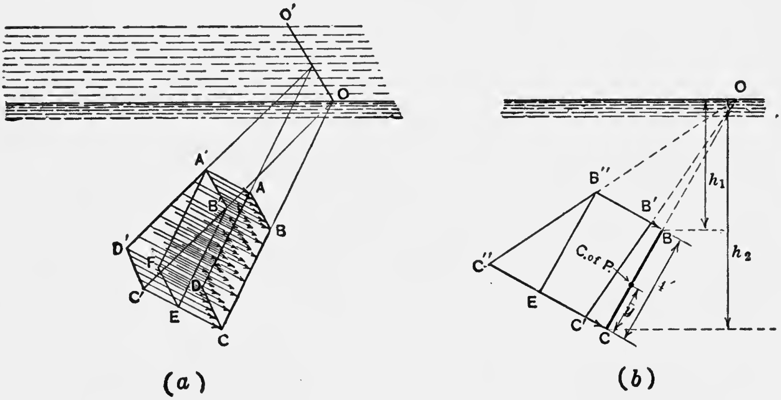

Fig. 20.

to $6BB'$ and $6CC'$, respectively, and therefore represent the areas $ABB'A'$ and $DCC'D'$. The total pressure acting on the surface $ABCD$ is therefore represented by the area of the pressure diagram $BCC''B''$ as it is similarly represented by the pressure volume $ABCDA'B'C'D'$. Also the resultant pressure on the surface acts through the center of gravity of the pressure area $BCC''B''$ just as it acts through the center of gravity of the pressure volume $ABCDA'B'C'D'$.

The trapezoid $BCC''B''$ may be divided into the rectangle $BB''EC$ and the triangle $B''C''E$, the locations of whose centers of gravity are known. By taking moments of each of these pressure areas about $C''C$ and dividing the sum of these moments by the

area of the trapezoid, the distance of the center of pressure from C is determined. Thus

$$BB''=w\times5\times6$$

and

$$CC''=w(5+2\sqrt{3})\times6,$$

therefore

$$C''E=w\times2\sqrt{3}\times6.$$

Taking moments about $C''C$,

$$\bar{y}=\frac{4\times w\times5\times6\times2+\frac{4}{2}\times w\times2\sqrt{3}\times6\times\frac{4}{3}}{4\times\frac{1}{2}[w\times5\times6+w(5+2\sqrt{3})\times6]}$$

$$=1.83 \text{ ft.}$$

Example (*c*), page 29, can also be solved by taking moments about BB'' as follows:

$$4F=4\times62.4\times5\times6\times2+\frac{4}{2}\times62.4\times2\sqrt{3}\times6\times\frac{3}{8}$$

from which

$$F=5470 \text{ lbs.}$$

For areas having a variable width, $OB''C''$ is not a straight line and the center of gravity of the pressure area is not so easily located. For such areas it will probably be easier to use the analytical method described in Art. 26.

28. Position of Center of Pressure with Respect to Center of Gravity.—If the intensity of pressure varies over any surface, the center of pressure is below the center of gravity. Consider the equation (see Art. 26),

$$\bar{y}=y'+\frac{k^2}{y'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Since $\frac{k^2}{y'}$ must always be positive, $\bar{y}$ must be greater than y'.

This may also be seen from Fig. 20. The center of pressure on $ABCD$ is the normal projection on that plane of the center of gravity of the pressure volume $ABCDA'B'C'D'$. Evidently this projection must fall below the center of gravity of $ABCD$ since it would fall at the center of gravity if the intensity of pressure on the surface were uniform, in which case the pressure volume would be $ABCDA'B'EF$.

It also appears from the above discussion and from a study of Fig. 20 that for any area the greater its depth below the surface of the liquid the more nearly will the center of pressure approach the center of gravity. The two coincide at an infinite depth.

In two cases the intensity of pressure is constant over the area and hence the center of pressure coincides with the center of gravity:

(*a*) When the surface is horizontal.

(*b*) When both sides of the area are completely submerged in liquids of the same density. As an illustration consider the gate *AB* (Fig. 21). Water stands h_1 feet above the top of the gate on one side and h_2 feet above it on the other side. The distribution of pressure on the left is represented by the trapezoid *ABMN* and on the right by the trapezoid *AHKB*. The triangle *GED* is similar to *CFG* and equal to *CE′P* by construction. The trapezoid of pressure *AHKB* is therefore balanced by the trapezoid *ONML*. The resultant intensity of pressure on the surface is therefore constant, as represented by the rectangle *OABL*, and the center of pressure must coincide with the center of gravity of the surface. This is true regardless of the shape of the surface. The resultant intensity of pressure equals wh, where h is the difference in elevation of water surfaces.

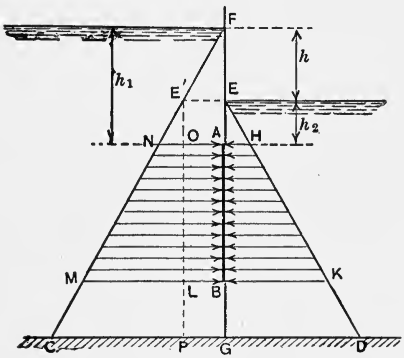

Fig. 21.

In this latter case it should be observed that it is the center of the *resultant* pressure that coincides with the center of gravity of the gate, since the center of gravity of either of the trapezoidal areas of pressure, considered alone, falls below the center of gravity of the gate.

29. Horizontal and Vertical Components of Pressure on any Surface.—It may be convenient to deal with the horizontal and vertical components of the pressure acting on a surface rather than with the resultant pressure.

Consider, for example, the water pressure acting on the curved

face, AB, of the dam shown in section in Fig. 22. The dam may have any length normal to the plane of the paper. Choosing the coordinate axes as shown, let BF represent the trace of a vertical plane normal to the XY plane. Consider the equilibrium of the volume of liquid whose cross-section, as shown in the figure, is ABF and whose ends are parallel with the XY plane and separated by a distance equal to the length of the dam. Since this volume of liquid is assumed to be in equilibrium, $\Sigma X=0$ and $\Sigma Y=0$.

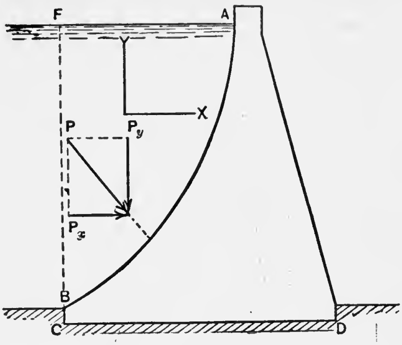

FIG. 22.

The only forces that have any components parallel with the X-axis are the X-components of the normal pressures acting on the surface AB and the normal pressure on the vertical plan BF. Since BF is the projection of the face AB on a vertical plane normal to the X-axis, it follows that the resultant of the X-components of the pressures on AB, or P_x, equals the normal pressure on the projection of AB on a vertical plane normal to the X-axis. As the demonstration holds true independently of the manner in which the X-axis is chosen, it may be stated in general that the component, along any horizontal axis, of the pressure on any area is equal to the normal pressure on that vertical projection of the area which is normal to the chosen axis.

In a similar manner consider the vertical forces acting on the volume of liquid whose cross-section is ABF (Fig. 22). The only vertical forces are the force of gravity, or the weight of the liquid, and the vertical components of the pressures on the surface AB, which forces must therefore be equal in magnitude. In other words, the vertical component of the pressure on any surface is equal to the weight of that volume of the liquid extending vertically from the surface to the free surface of the liquid. If the pressure were acting upward on the surface, its magnitude, as will be shown later (Art. 30), would be equal to the weight of that volume of the liquid that *would* extend from the surface to the free surface of the liquid. The pressures considered in this article are relative

pressures, since obviously atmospheric pressure is acting on both sides of the dam and the resultant effect is zero.

Examples: (*a*) What will be the resultant pressure on the base BC of the masonry dam subjected to water pressure, as shown in Fig. 23, and where will this resultant intersect the base?

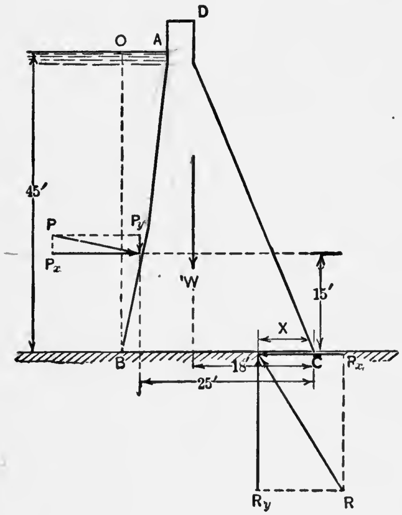

FIG. 23.

The following numerical values are given. Area of section $ABCD = 600$ sq. ft. Area of water section $OAB = 200$ sq. ft. Weight of masonry $= 150$ lbs. per cubic foot. Linear dimensions are shown in figure.

A section of dam 1 ft. long will be considered to be in equilibrium under the action of the following forces:

W = the total weight of the section, acting through the center of gravity of the cross-section $ABCD$;
P = resultant hydrostatic pressure acting on the face AB;
R = reaction between the earth and the base, BC, of the dam.

This reaction must necessarily be equal, opposite and colinear with the resultant of W and P.

Since the dam is in equilibrium when subjected to the above forces, the fundamental principles of equilibrium may be applied—that is,

$$\Sigma X = 0,\ \Sigma Y = 0 \quad \text{and} \quad \Sigma M = 0.$$

For $\Sigma X = 0$; $R_x = P_x$ and for the data given

$$P_x = 62.4 \times \tfrac{45}{2} \times 45 = 63{,}200 \text{ lbs.}$$

For $\Sigma Y = 0$; $R_y = P_y + W$ and for the data given

$$P_y = 200 \times 62.4 = 12{,}480 \text{ lbs.}$$

$$W = 600 \times 150 = 90{,}000 \text{ lbs.}$$

$$R_y = 12{,}480 + 90{,}000 = 102{,}480 \text{ lbs.}$$

The total resultant pressure on the base is

$$R = \sqrt{63{,}200^2 + 102{,}480^2} = 122{,}500 \text{ lbs.}$$

For $\Sigma M = 0$, taking moments about an axis through C, normal to the plane of the section, X being the distance from C at which the resultant, R, intersects the base of the dam,

$$R_y X + \tfrac{45}{3} P_x - 25 P_y - 18W = 0.$$

Substituting the numerical values of R_y, P_x, W and P_y

$$102{,}480X + 63{,}200 \times \tfrac{45}{3} - 12{,}480 \times 25 - 90{,}000 \times 18 = 0$$

and reducing

$$X = 8.6 \text{ ft.}$$

Thus the resultant pressure of 122,500 lbs. per linear foot of dam intersects the base 8.6 ft. from the toe of the dam.

(*b*) Determine the tensile stress in the walls of a 24-in. pipe carrying water under a head of 100 ft.

In a case like this where the head is relatively large compared to the diameter of the pipe, it is customary to consider that the intensity of pressure is uniform throughout the pipe.

A cross-section of the pipe is shown in Fig. 24. Consider a semicircular segment, AB, of unit length, held in equilibrium by the two forces T. Evidently T is the tensile stress in the wall of the pipe, and if the intensity of pressure is assumed to be constant, T is constant at all points in the section. The sum of the horizontal components of the normal pressures acting on the semicircular segment is equal to the normal pressure on the vertical projection of this segment (Art. 29). Calling this normal pressure P, since $\Sigma H = 0$,

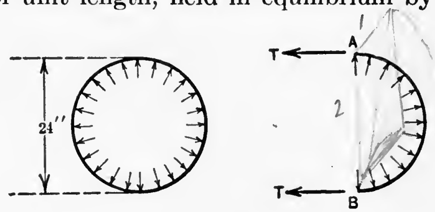

FIG. 24.

$$2T = P = whA = 62.4 \times 100 \times 2 = 12{,}480 \text{ lbs.}$$

and

$$T = 6{,}240 \text{ lbs.}$$

The required thickness of a steel pipe, using a safe working stress of 16,000 lbs. per square inch is

$$t=\frac{6240}{16000\times 12}=0.0325 \text{ in.}$$

or a little more than $\frac{1}{32}$ in.

PROBLEMS

1. A vertical rectangular gate is 4 ft. wide and 6 ft. high. Its upper edge is horizontal and on the water surface. What is the total pressure on the gate and where is the center of pressure?

2. Solve Problem 1 if the water surface is 5 ft. above the top of the gate, other conditions remaining the same.

3. Solve Problem 2 if the plane of the gate makes an angle of 30° with the vertical, other conditions remaining unchanged.

4. A cubical box, 24 in. on each edge, has its base horizontal and is half filled with water. One of the sides is held in position by means of four screws, one at each corner. Find the tension in each screw due to the water pressure.

5. A vertical, triangular gate has a horizontal base 4 ft. long, 3 ft. below the vertex and 5 ft. below the water surface. What is the total pressure on the gate and where is the center of pressure?

6. A vertical, triangular gate has a horizontal base 3 ft. long and 2 ft. below the water surface. The vertex of the gate is 4 ft. below the base. What force normal to the gate must be applied at its vertex to open the gate?

7. A triangular gate having a horizontal base 4 ft. long and an altitude of 6 ft. is inclined 45° from the vertical with the vertex pointing upward. The base of the gate is 8 ft. below the water surface. What normal force must be applied at the vertex of the gate to open it?

8. A cylindrical tank, having a vertical axis, is 6 ft. in diameter and 10 ft. high. Its sides are held in position by means of two steel hoops, one at the top and one at the bottom. What is the tensile stress in each hoop when the tank is filled with water?

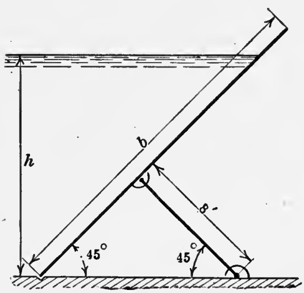

Fig. 25.

9. What is the greatest height, h, to which the water can rise without causing the dam shown in Fig. 25 to collapse? Assume b to be so great that water will not flow over the top of the dam.

10. If b in the figure is 20 ft., find the least value of h at which the dam will collapse.

11. A vertical, triangular gate has a horizontal base 8 ft. long and 6 ft. below the water surface. Its vertex is 2 ft. above the water surface. What normal force must be applied at the vertex to open the gate?

12. A masonry dam of trapezoidal cross-section, with one face vertical,

has a thickness of 2 ft. at the top and 10 ft. at the bottom. It is 22 ft. high and has a horizontal base. The vertical face is subjected to water pressure, the water standing 15 ft. above the base. The weight of the masonry is 150 lbs. per cubic foot. Where will the resultant pressure intersect the base?

13. In Problem 12 what would be the depth of water when the resultant pressure intersects the base at the outer edge of the middle third, or $1\frac{2}{3}$ ft. from the middle of the base?

14. A vertical triangular surface has a horizontal base of 4 ft. and an altitude of 9 ft., the vertex being below the base. If the center of pressure is 6 in. below the center of gravity, how far is the base below the water surface?

15. Water stands 40 ft. above the top of a vertical gate which is 6 ft. square and weighs 3000 lbs. What vertical lift will be required to open the gate if the coefficient of friction between the gate and the guides is 0.3?

16. On one side, water stands level with the top of a vertical, rectangular gate 4 ft. wide and 6 ft. high, hinged at the top; on the other side water stands 3 ft. below the top. What force applied at the bottom of the gate, at an angle of 45° with the vertical, is required to open the gate?

17. A vertical, trapezoidal gate in the face of a dam has a horizontal base 8 ft. below the water surface. The gate has a width of 6 ft. at the bottom and 3 ft. at the top, and is 4 ft. high. Determine the total pressure on the gate and the distance from the water surface to the center of pressure.

18. Determine the total pressure and the position of the center of pressure on a vertical, circular surface 3 ft. in diameter, the center of which is 4 ft. below the water surface.

19. A 6-in. pipe line in which there is a 90° bend contains water under a gage pressure of 450 lbs. per square inch. Assuming that the pressure is uniform throughout the pipe and that the water is not in motion, find the total longitudinal stress in joints at either end of the bend.

CHAPTER IV

IMMERSED AND FLOATING BODIES

30. Principle of Archimedes.—Any body immersed in a liquid is subjected to a buoyant force equal to the weight of liquid displaced. This is known as the principle of Archimedes. It may be proved in the following manner.

The submerged body $ABCD$ (Fig. 26) is referred to the coordinate axes X, Y and Z. Consider the small horizontal prism a_1a_2, parallel to the X-axis, to have a cross-sectional area dA. The X-component of the normal force acting on a_1 must be equal and opposite to the same force acting on a_2, each being equal to $wh\,dA$. There is, therefore, no tendency for this prism to move in a direction parallel to the X-axis. Since the same reasoning may be applied to every other prism parallel to a_1a_2 it follows that there is no tendency for the body as a whole to move in this direction. The same reasoning applies to movement parallel to the Z-axis or to any other axis in a horizontal plane. If, therefore, there is any tendency for the body to move it must be in a vertical direction.

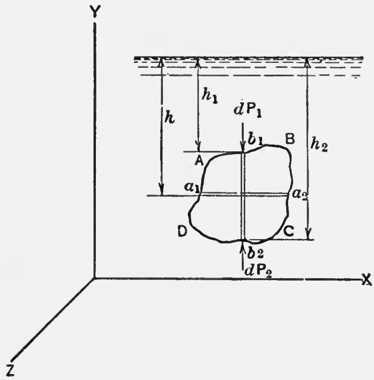

Fig. 26.

Consider now the Y-components of the hydrostatic pressures acting on the ends of any vertical prism b_1b_2 having a cross-sectional area, dA, so small that the intensity of pressure on either end of the prism may be considered constant. The resultant of these pressures will be the difference between dP_1, the vertical component of the normal pressure at b_1 equal to $wh_1\,dA$, acting downward and dP_2, the corresponding force acting at b_2, equal to $wh_2\,dA$,

acting upward. The resultant pressure will be upward and equal to $w(h_2-h_1)\,dA$; but $(h_2-h_1)\,dA$ is the volume of the elementary prism which, multiplied by w, gives the weight of the displaced liquid. Since the entire body, $ABCD$, is made up of an infinite number of such prisms, it follows that the resultant hydrostatic pressure on the body will be a buoyant force equal in magnitude to the weight of the displaced liquid.

If the weight of the body is greater than the buoyant force of the liquid the body will sink. On the other hand, if the weight of the body is less than the buoyant force, the body will float on the surface, displacing a volume of liquid having a weight equal to that of the body.

31. Center of Buoyancy.—$ABCD$ (Fig. 27) represents a floating body. From the principles of the preceding article, the buoyant force acting on any elementary area of the submerged surface must be equal to the weight of the vertical prism of displaced liquid directly above it. Since the weight of each prism is directly proportional to its volume, the center of gravity of all these buoyant forces, or the *center of buoyancy*, must coincide with the center of gravity of the displaced liquid.

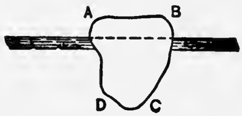

Fig. 27.

32. Stability of Floating Bodies.—Any floating body is subjected to two systems of parallel forces; the downward force of gravity acting on each of the particles that goes to make up the body and the buoyant force of the liquid acting upward on the various elements of the submerged surface.

In order that the body may be in equilibrium the resultants of these two systems of forces must be colinear, equal and opposite. Hence the center of buoyancy and the center of gravity of the floating body must lie in the same vertical line.

Fig. 28 (a) shows the cross-section of a ship floating in an upright position, the axis of symmetry being vertical. For this position the center of buoyancy lies on the axis of symmetry at B_0 which is the center of gravity of the area ACL. The center of gravity of the ship is assumed to be at G. If, from any cause, such as wind or wave action, the ship is made to heel through an angle θ, as shown in Fig. 28 (b), the center of gravity of the ship and cargo remaining unchanged, the center of buoyancy will shift to a new position, B, which is the center of gravity of the area

$A'C'L$. The buoyant force F, acting upward through B, and the weight of the ship W, acting downward through G, constitute a couple which resists further overturning and tends to restore the ship to its original upright position. In all cases, if the vertical line through the center of buoyancy intersects the inclined axis of symmetry at a point M above the center of gravity, the two forces F and W must produce a *righting moment.* If, however, M lies below G an *overturning moment* is produced. The point M is known as the *metacenter* and its distance, GM, from the center of gravity of the ship, is termed the *metacentric height.* The value of the metacentric height is a measure of the stability of the ship.

For angles of inclination not greater than 10° or 15° the

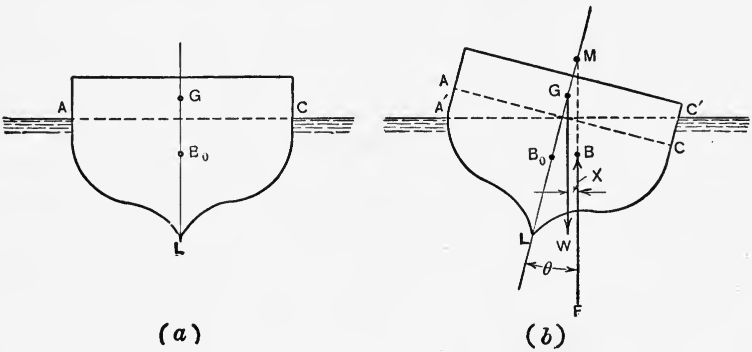

Fig. 28.

position of M does not change materially, and for small angles of heel the metacentric height may be considered constant. For greater inclinations, however, the metacentric height varies to a greater extent with the angle of heel.

33. Determination of Metacentric Height.—Fig. 29 illustrates a ship having a displacement volume, V. When the ship is tilted through the angle θ the wedge AOA' emerges from the water while the wedge $C'OC$ is immersed. If the sides AA' and $C'C$ are parallel, these wedges must be similar and of equal volume, v, since the same volume of water is displaced by the ship whether in an inclined or upright position. The wedges therefore will have the same length and the water lines AC and $A'C'$ must intersect on the axis of symmetry at O.

When the ship floats in an upright position a buoyant force

F, equal to wv, acts upward through K, the center of gravity of the wedge AOA'. In the inclined position this force no longer acts, but an equal force F' acts at K', the center of gravity of the wedge $C'OC$. It may be considered that a downward force F'', equal to F, has been introduced, the resultant of F'' and F being zero. A righting couple has therefore been introduced equal to wvL, L being the horizontal distance between the centers of gravity of the wedges.

Because of the shifting of the force F from K to K' the line of

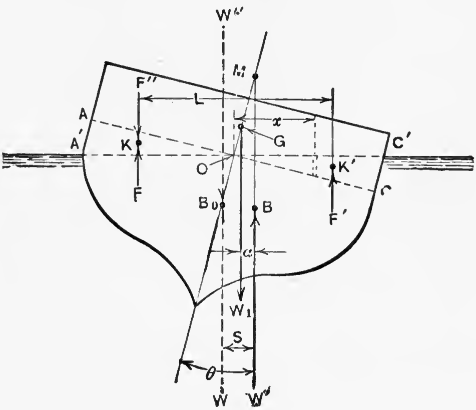

Fig. 29.

action of the buoyant force W acting on the entire ship is shifted from B_0 to B, a horizontal distance S such that

$$wVS = wvL. \qquad (1)$$

Consider now a small vertical prism of the wedge $C'OC$, at a distance x from O, having a cross-sectional area dA. The buoyant force produced by this immersed prism is $wx \tan \theta \, dA$, and the moment of this force about O is $wx^2 \tan \theta \, dA$.

The sum of all of these moments for both wedges must be equal to wvL or

$$w \tan \theta \int x^2 \, dA = wvL = wVS.$$

But $S = MB_0 \sin \theta$, and for small angles, since the sine is very nearly equal to the tangent,

$$\int x^2 \, dA = V(MB_0) = V(GM \pm GB_0),$$

and since $\int x^2\,dA$ is the moment of inertia, I, of the water-line section about the longitudinal axis through O,

$$GM=\frac{I}{V}\pm GB_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the sign in the last expression being positive if M falls below G and negative if above.

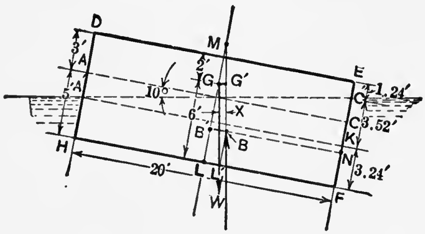

FIG. 30.

Example.—Find the metacentric height of the rectangular scow shown in Fig. 30.

The scow is 40 ft. long, 20 ft. wide and 8 ft. deep. It has a draft of 5 ft. when floating in an upright position. The center of gravity of the scow is on the axis of symmetry, 1 ft. above the water surface. The angle of heel is 10°.

The problem may be solved by substituting values in formula (2), but the following method may be used conveniently for shapes such that the center of buoyancy can be readily found.

Since the center of buoyancy, B, is at the center of gravity of $HA'C'F$, its position may be found by taking moments about the axes HF and EF. Before taking moments the distances KC' and KF must be determined.

$$KC'=20\times \tan 10°=3.52,$$

$$KF=5-\frac{3.52}{2}=3.24.$$

Taking moments about HF

$$(5\times 20)BL'=\left(3.24\times 20\times\frac{3.24}{2}\right)+\frac{3.52}{2}\times 20\left(3.24+\frac{3.52}{3}\right),$$

$$BL'=2.60 \text{ ft.}$$

Taking moments about EF

$$(5\times 20)BN=\left(3.24\times 20\times\frac{20}{2}\right)+\left(\frac{3.52}{2}\times 20\times\frac{20}{3}\right),$$

$$BN=8.83 \text{ ft.}$$

The distance of B from the inclined axis of symmetry is

$$BB' = 10 - 8.83 = 1.17 \text{ ft.}$$

From similar triangles

$$\frac{MB'}{BB'} = \frac{A'K}{C'K} \quad \text{or} \quad MB' = \frac{1.17 \times 20}{3.52} = 6.65 \text{ ft.}$$

The metacentric height is

$$GM = MB' + B'L - GL$$

$$= 6.65 + 2.60 - 6 = 3.25 \text{ ft.}$$

The righting moment is

$$Wx = 5 \times 20 \times 40 \times 62.4 \times 3.25 \sin 10°$$

$$= 140{,}900 \text{ ft.-lbs.}$$

PROBLEMS

1. A rectangular scow 15 ft. by 32 ft., having vertical sides, weighs 40 tons (80,000 lbs.). What is its draft?

2. If a rectangular scow 18 ft. by 40 ft. has a draft of 5 ft. what is its weight?

3. A cubic foot of ice (sp. gr. = 0.90) floats freely in a vessel containing water whose temperature is 32° F. When the ice melts, will the water level in the vessel rise, lower or remain stationary? Explain why.

4. A ship of 4000 tons displacement floats with its axis of symmetry vertical when a weight of 50 tons is midship. Moving the weight 10 ft. towards one side of the deck causes a plumb bob, suspended at the end of a string 12 ft. long, to move 9 in. Find the metacentric height.

5. A rectangular scow 30 ft. wide, 50 ft. long, and 12 ft. high has a draft of 8 ft. Its center of gravity is 9 ft. above the bottom of the scow. If the scow is tilted until one side is just submerged, determine:

(*a*) The position of the center of buoyancy.
(*b*) The metacentric height.
(*c*) The righting couple, or the overturning couple.

6. In Problem 5, what would be the height of the scow (all other data remaining unchanged) if, with one side just submerged, the scow would be in unstable equilibrium?

7. A box, 1 ft. square and 6 ft. high, has its upper end closed and lower end open. By submerging it vertically with the open end down what is the greatest weight the box can sustain without sinking?

8. In Problem 7, what weight would hold the box in equilibrium with the upper end submerged 10 ft. below the surface?

9. A solid block of wood (sp. gr. = 0.6) in the shape of a right cone has a

base whose diameter is 12 in. and an altitude of 18 in. In what position will this block float in water when it is in stable equilibrium?

10. A solid block of wood (sp. gr. $=0.6$) in the shape of a right cylinder has a diameter of 12 in. and a length of 15 in. Determine the position in which this block will float in water when in stable equilibrium.

CHAPTER V

RELATIVE EQUILIBRIUM OF LIQUIDS

34. Relative Equilibrium Defined.—In the preceding chapters liquids have been assumed to be in equilibrium and at rest with respect both to the earth and to the containing vessel. The present chapter treats of the condition where every particle of a liquid is at rest with respect to every other particle and to the containing vessel, but the whole mass, including the vessel, has a uniformly accelerated motion with respect to the earth. The liquid is then in equilibrium and at rest with respect to the vessel, but it is neither in equilibrium nor at rest with respect to the earth. In this condition a liquid is said to be in *relative equilibrium.* Since there is no motion of the liquid with respect to the vessel and no movement between the water particles themselves there can be no friction.

Hydrokinetics, which is treated in the following chapters, deals with the condition in which water particles are in motion with respect to the earth and also with respect to each other. In this case the retarding effects of friction must be considered.

Relative equilibrium may be considered as an intermediate state between hydrostatics and hydrokinetics. Two cases of relative equilibrium will be discussed.

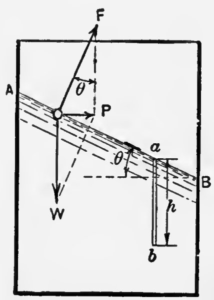

FIG. 31.

35. Vessel Moving with Constant Linear Acceleration.—If a vessel partly filled with any liquid moves horizontally along a straight line with a constant acceleration, j, the surface of the liquid will assume an angle θ with the horizontal as shown in Fig. 31. To determine the value of θ for any value of j, consider the forces acting on a small mass of liquid, M, at any point O on the surface. This mass is moving with a constant horizontal acceleration, j, and the force producing the acceleration is the resultant of all

the other forces acting upon the mass. These forces are the force of gravity, W, acting vertically downward and the pressure of all the contiguous particles of the liquid. The resultant, F, of these forces must be normal to the free surface AB. Since force equals mass times acceleration,

$$P=Mj=\frac{Wj}{g}, \quad \quad (1)$$

and from the figure

$$P=W\tan\theta. \quad \quad (2)$$

Solving these two equations simultaneously,

$$\tan\theta=\frac{j}{g}, \quad \quad (3)$$

which gives the slope that the surface, AB, will assume for any constant acceleration of the vessel.

Since O was assumed to be anywhere on the surface and the values of j and g are the same for all points, it follows that $\tan\theta$ is constant at all points on the surface or, in other words, AB is a straight line.

The same value of θ will hold for a vessel moving to the right with a positive acceleration as for a vessel moving to the left with a negative acceleration or a retardation.

To determine the intensity of pressure at any point b, at a depth h below the free surface, consider the vertical forces acting on a vertical prism ab (Fig. 31). Since there is no acceleration vertically the only forces acting are atmospheric pressure at a, gravity, and the upward pressure on the base of the prism at b. Hence, if the cross-sectional area is dA,

$$p_b\,dA=wh\,dA+p_a\,dA, \quad \quad (4)$$

or

$$p_b=wh+p_a, \quad \quad (5)$$

or, neglecting atmospheric pressure which acts throughout,

$$p_b=wh. \quad \quad (6)$$

Therefore, in a body of liquid moving with a horizontal acceleration the relative pressure at any point is that due to the head of liquid directly over the point, as in hydrostatics. In this case, however, it is evident that all points of equal pressure lie in an inclined plane parallel with the surface of the liquid.

In equation (3) if j were zero, tan θ would equal zero; or, in other words, if the vessel were moving with a constant velocity the surface of the liquid would be horizontal. Also if the acceleration were vertically upward, the surface would obviously be horizontal.

To determine the relative pressure at any point, b, in a vessel with an acceleration upward, consider the forces acting on a vertical prism of liquid ab of height h and cross-sectional area dA (Fig. 32). The force, P, producing the acceleration is the resultant of all the forces acting on the prism, consisting of gravity equal to $wh\,dA$, acting downward and the static pressure on the lower end of the filament at b, equal to $pt\,dA$, acting upward. Therefore

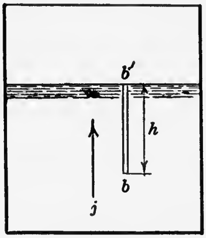

FIG. 32.

$$P=p_b dA-wh\,dA=Mj=\frac{wh\,dA}{g}j,$$

From which

$$p_b=wh+wh\frac{j}{g}. \quad \ldots\ldots\ldots\ldots (7)$$

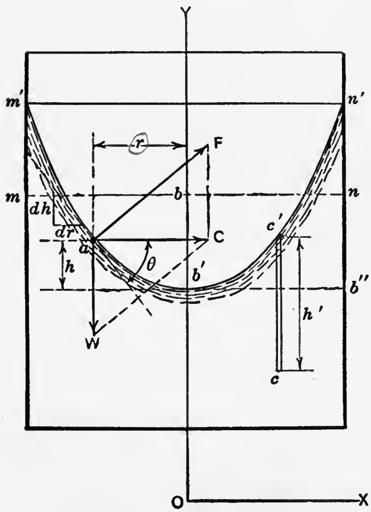

FIG. 33.

This shows that the intensity of pressure at any point within a liquid contained in a vessel having an upward acceleration, j, is greater than the static pressure by an amount equal to $wh\frac{j}{g}$. Evidently, if the acceleration were downward, the sign of the last term in the above expression would become negative, and if $j=g$, $p_b=0$. In other words, if a vessel containing any liquid falls freely in a vacuum, so as not to be retarded by air friction, the pressure will be zero at all points throughout the vessel.

36. Vessel Rotating about a Vertical Axis.—When the vessel shown in Fig. 33 is at rest, the

surface of the liquid is horizontal and at mn. $m'b'n'$ represents the form of surface resulting from rotating the vessel with a constant angular velocity ω radians per second about its vertical axis OY.

Consider the forces acting on a small mass of liquid, M, at a, distant r from the axis OY.

Since this mass has a uniform circular motion it is subjected to a centripetal force,

$$C = M\omega^2 r, \quad \dots \dots \dots \quad (8)$$

which force produces an acceleration directed toward the center of rotation and is the resultant of all the other forces acting on the mass. These other forces are the force of gravity, $W = Mg$, acting vertically downward, and the pressure exerted by the adjacent particles of the liquid. The resultant, F, of this liquid pressure must be normal to the free surface of the liquid at a.

Designating by θ the angle between the tangent at a and the horizontal,

$$\tan\theta = \frac{dh}{dr} = \frac{C}{W} = \frac{M\omega^2 r}{Mg} = \frac{\omega^2 r}{g},$$

or

$$dh = \frac{\omega^2 r}{g} dr,$$

which, when integrated becomes,

$$h = \frac{\omega^2 r^2}{2g}. \quad \dots \dots \dots \quad (9)$$

The constant of integration equals zero, since when r equals zero h also equals zero.

Since h and r are the only variables this is the equation of a parabola, and the liquid surface is a paraboloid of revolution about the Y-axis. As the volume of a paraboloid is equal to one-half that of the circumscribed cylinder and since the volume of liquid within the vessel has not been changed,

$$b'b = \tfrac{1}{2}b''n' = nn'.$$

The linear velocity at a is

$$v = \omega r. \quad \dots \dots \dots \quad (10)$$

Substituting v for ωr in equation (9)

$$h=\frac{v^2}{2g}. \qquad (11)$$

Expressed in words, this means that any point on the surface of the liquid will rise above the elevation of greatest depression a height equal to the velocity head (see Art. 43) at that point.

To determine the relative pressure at any point c at a depth h' vertically below the surface at c' consider the vertical forces acting on the prism cc', having a cross-sectional area dA. As this prism has no vertical acceleration, $\Sigma y=0$ and

$$p_c dA = wh' \, dA$$

or

$$p_c = wh'. \qquad (12)$$

That is, the relative pressure at any point is that due to the head of liquid directly over the point, as in hydrostatics. Therefore the distribution of pressure on the bottom of the vessel is represented graphically by the vertical ordinates to the curve $m'b'n'$. It also follows that the total pressure on the sides of the vessel is the same as though the vessel were filled to the level $m'n'$ and were not rotating.

PROBLEMS

1. A vessel containing water moves horizontally along a straight line with a constant velocity of 10 ft. per second. What is the form of its water surface?

2. A vessel partly filled with water moves horizontally with a constant, linear acceleration of 10 ft. per second per second. What is the form of its water surface?

3. An open cylindrical vessel, 2 ft. in diameter, 3 ft. high and two-thirds filled with water, rotates about its vertical axis with a constant speed of 90 R.P.M. Determine:

(*a*) The depth of water at the center of the vessel.
(*b*) The total pressure on the cylindrical walls.
(*c*) The total pressure on the bottom of the vessel.

4. In Problem 3, what is the greatest speed in revolutions per minute that the vessel can have without causing any water to spill over the sides?

5. In Problem 3, what speed in revolutions per minute must the vessel have in order that the depth at the center will be zero?

6. In Problem 3, what speed in revolutions per minute must the vessel have in order that there may be no water within 6 in. of the vertical axis?

7. If a closed cylindrical vessel, 2 ft. in diameter, 3 ft. high and completely filled with water, rotates about its vertical axis with a speed of 240 R.P.M., determine the intensity of pressure at the following points:

(*a*) At the circumference, just under the cover.
(*b*) At the axis, just under the cover.
(*c*) At the circumference, on the bottom.
(*d*) At the axis, on the bottom.

CHAPTER VI

PRINCIPLES OF HYDROKINETICS

37. Introductory.—The principles relating to the behavior of water or other liquids at rest are based upon certain definite laws which hold rigidly in practice. In solving problems involving these principles it is possible to proceed by purely rational methods, the results obtained being free from any doubt or ambiguity. Calculations are based upon a few natural principles which are universally true and simple enough to permit of easy application. In all problems ordinarily encountered in hydrostatics, after the unit weight of the liquid has been determined, no other experimental data are required.

A liquid in motion, however, presents an entirely different condition. Though the motion undoubtedly takes place in accordance with fixed laws, the nature of these laws and the influence of the surrounding conditions upon them are very complex and probably incapable of being expressed in any exact mathematical form.

Friction and viscosity affect the laws of hydrokinetics in a varying degree for different liquids. Since water is the most common liquid with which the engineer has to deal and since, as a result, more is known about the laws relating to the flow of this liquid, the following treatise on hydrokinetics applies only to *water*. The fundamental principles discussed hold true for all liquids, but the working formulas would necessarily have to be modified for each different kind of liquid.

A clearer conception of the underlying principles of hydrokinetics is made possible by the assumption of certain ideal conditions. This also permits of the establishment of a few basic laws which may be expressed as fundamental formulas. These assumed conditions, however, vary widely from those which actually exist and working formulas based upon them must invariably be modified by experimental coefficients.

A formula with its empirical coefficients included, which requires only that numerical values be affixed to the coefficients to make it adaptable to the solution of problems, is referred to as a base formula. Many formulas used in hydrokinetics differ so widely from the fundamental form that they have little if any claim to a rational basis.

During the last two centuries many hundreds of experiments on flowing water have been performed. These experiments have covered a wide range of conditions, and the data obtained from them make possible the modern science of hydrokinetics.

38. Friction.—There can be no motion between two substances in contact without friction. This principle applies to liquids and gases as well as solids. Water flowing in any conduit encounters friction with the surfaces with which it comes in contact. There is also friction between the moving particles of water themselves, commonly called viscosity (Art. 6). The free surface of water flowing in an open channel encounters the resistance of the air and also the greater resistance of the surface skin which results from surface tension (Art. 7).

The amount of frictional resistance offered by any surface increases with the degree of roughness of the surface. The resistance which results from viscosity decreases as the temperature of the water increases. The influence of friction and viscosity on the flow of water must be determined experimentally.

To overcome frictional resistance requires an expenditure of energy. The expended energy is transformed into heat. After being so transformed it cannot, through the ordinary processes of nature, be reconverted into any of the useful forms of energy contained in flowing water and is therefore often referred to as *lost* energy.

39. Discharge.—The rate of flow or the volume of water passing a cross-section of a stream in unit time is called the *discharge.* The symbol Q will be used to designate the discharge in cubic feet per second. Other units of discharge, such as cubic feet per minute, gallons per minute or gallons per day are sometimes employed for special purposes.

If a uniform velocity at all points in the cross-section of a stream were possible there would be passing any cross-section every second a prism of water having a base equal to the cross-sectional area of the stream and a length equal to the velocity. Because,

however, of the varying effects of friction and viscosity, the different filaments of water move with different velocities. For this reason it is common in hydraulics to deal with mean velocities. If v is the mean velocity in feet per second past any cross-section, and a is the cross-sectional area in square feet,

$$Q=av \qquad (1)$$

and

$$v=\frac{Q}{a}. \qquad (2)$$

These simple formulas are of fundamental importance.

40. Steady Flow and Uniform Flow.—If the same quantity of water passes any cross-section of a stream during equal successive intervals of time the flow is said to be *steady*. If the quantity of water passing any cross-section changes during successive intervals of time the flow is said to be *unsteady*. If not otherwise stated, the condition of steady flow will be assumed. The fundamental principles and formulas based upon steady flow do not generally hold for unsteady flow. Problems most commonly encountered in practice deal only with steady flow.

If in any reach of a stream the velocity at every cross-section is the same at any instant the flow is said to be *uniform*. This condition requires a stream of uniform cross-section. If the cross-section is not uniform throughout the reach, in the portions of the reach where velocity changes occur, the flow is *non-uniform*.

Thus, uniform flow implies instantaneous similarity of conditions at successive cross-sections, whereas steady flow involves permanency of conditions at any particular cross-section.

41. Continuity of Discharge.—When, at any instant, the discharge is the same past every cross-section, it is said to be continuous, or there is *continuity of discharge*. The term *continuity of flow* is also used to express this condition. Letting Q, a and v represent, respectively, discharge, area and mean velocity with similar subscripts applying to the same cross-section, continuity of discharge exists when

$$Q=a_1v_1=a_2v_2=a_3v_3, \text{ etc.} \qquad (3)$$

Continuity of discharge may be illustrated by assuming water to be turned into a canal. At first there will be a greater volume of water flowing near the entrance than at sections farther down.

Under such conditions the discharge is not continuous. Ultimately, however, if the supply of water is constant and assuming no losses from seepage or evaporation, there will be the same quantity of water flowing past all sections of the canal, and the condition of continuity of discharge will exist in the entire canal regardless of whether or not all reaches of the canal have the same cross-sectional area. In a pipe flowing full, even though the pipe is made up of several diameters, the discharge is continuous.

42. Stream Line and Turbulent Motion.—Flowing water is said to have stream-line motion if each particle follows the same path as was followed by every preceding particle that occupied the same position. If stream-line motion exists within a conduit having parallel sides the paths of the water particles are parallel to the sides of the conduit and to each other.

Water flows with stream-line motion only at very low velocities,

(a) (b)

Fig. 34.

excepting in very small pipes where such motion may exist at quite high velocities (Art. 90). Under practically all conditions encountered in the field of engineering, the motion is *turbulent*, the water particles moving without any regularity and not in accordance with any known laws. In Fig. 34 (*a*) and (*b*) represent, respectively, stream-line and turbulent motion.

Friction and viscosity affect the flow of water whether the motion be stream line or turbulent, but the effects produced in the two cases are in accordance with different laws (Art. 90).

43. Energy and Head.—Since the principles of energy are applied in the derivation of fundamental hydraulic formulas, an explanation of such principles as will be used is here introduced.

Energy is defined as ability to do work. Where the English system of units is employed, both energy and work are measured in foot-pounds. The two forms of energy commonly recognized are kinetic energy and potential energy.

Kinetic energy is the ability of a mass to do work by virtue of its

velocity. Where v is the velocity in feet per second and M, the mass in gravitational units, is equal to W/g, the kinetic energy of any mass is expressed by the equation

$$KE=\frac{Mv^2}{2}=\frac{Wv^2}{2g}, \qquad (4)$$

which reduces to $\frac{v^2}{2g}$ for a weight of unity. The expression $\frac{v^2}{2g}$ is of the form

$$\frac{\text{feet per second}\times\text{feet per second}}{\text{feet per second per second}}=\text{feet},$$

and it therefore represents a linear quantity expressed in feet. It is the distance which a body must fall in a vacuum to acquire the velocity v. When applied to flowing water it is called the *velocity head.* Although representing a linear quantity, the velocity head is directly proportional to the kinetic energy of any mass having a velocity v and is equal to the kinetic energy of one pound of any substance moving with a velocity v.

Potential energy is latent or potential ability to do work. Water manifests this ability in two ways:

(*a*) By virtue of its position or elevation with respect to some arbitrarily selected horizontal datum plane, considered in connection with the action of gravity. This may be called energy of position, energy of elevation or gravitational energy.

(*b*) By virtue of pressure produced by the action of gravity, or by the application of some external force, on the water. This may be called pressure energy.

Energy of position may be explained by considering a mass having a weight of W pounds whose elevation above any horizontal datum plane is h feet. With respect to this plane the mass has Wh foot-pounds of energy. A mass weighing one pound will have h foot-pounds of energy. If a mass weighing one pound is placed h feet below the datum plane, its energy with respect to the plane will be $-h$ foot-pounds, being negative because this amount of energy will have to be exerted upon the mass to raise it to the datum plane against the action of gravity. Here again the expression for energy, in this case h, represents a linear quantity which is the elevation head of the mass, but it should be kept clearly in mind that it is also the energy expressed in foot-pounds

contained in one pound of water by virtue of its position with respect to the datum plane.

It thus appears that the amount of energy of position possessed by a mass depends upon the elevation of the datum plane. In any particular problem, however, all masses should be referred to the same plane. This gives the relative amounts of energy contained in different masses or the relative amounts of energy in the same mass in different positions, which is all that is usually required.

The action of *pressure energy* is illustrated by the piston and cylinder arrangement shown in Fig. 35, which is operated entirely by water under a gage pressure of p pounds per square foot. The area of the piston is A square feet. The cylinder is supplied with water through the valve R and may be emptied through the valve S.

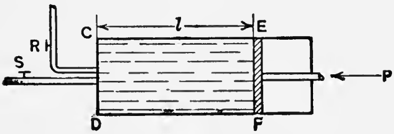

Fig. 35.

At the beginning of the stroke the piston is at CD, the valve S is closed and R is open. Water enters the cylinder and slowly drives the piston to the right against the force P. Neglecting friction, the amount of work done on the piston while moving through the distance l feet is $Pl=pAl$ foot-pounds. If R is now closed and S opened, again neglecting friction, the piston may be moved back to its original position without any work being done upon it. The quantity of water required to do the work, pAl foot-pounds, is Al cubic feet and its weight is wAl pounds. The amount of work done per pound of water is therefore

$$\frac{pAl}{wAl}=\frac{p}{w} \text{ foot-pounds.}$$

Since this work is done entirely at the expense of pressure energy and while the gage pressure is being reduced from p to zero, the amount of pressure energy per pound of water is $\frac{p}{w}$ foot-pounds.

It has been shown in Art. 17 that $\frac{p}{w}$ represents pressure head or a linear quantity. If pressure head is expressed in feet of water column, it will also represent foot-pounds of energy per pound of water as has been shown to be the case for velocity

head and elevation head. There are other forms of energy, such as heat energy and electrical energy, which have no direct bearing on the laws governing flowing water.

The three forms of energy which water may have are illustrated in Fig. 36. At A water is moving with a velocity v. The kinetic energy of a pound of water at A is $v^2/2g$, the pressure energy is

$$\frac{p}{w}=h$$

and the energy of position referred to the datum plane MN is z. Thus, with respect to the plane MN the *total energy per pound of water* at any point A, expressed in foot-pounds, is

$$E=\frac{v^2}{2g}+\frac{p}{w}+z. \qquad (5)$$

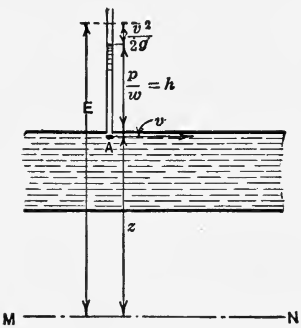

Fig. 36.

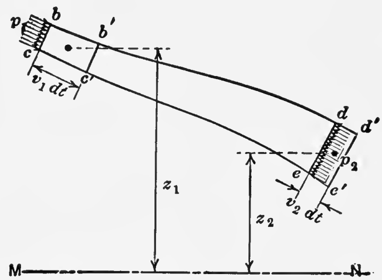

Fig. 37.

44. Bernoulli's Theorem.—In 1738 Daniel Bernoulli, an Italian engineer, demonstrated that in any stream flowing steadily without friction the total energy contained in a given mass of water is the same at every point in its path of flow. In other words kinetic energy, pressure energy and energy of position may each be converted into either of the other two forms, theoretically without loss. Thus if there is a reduction in the amount of energy contained in any one form there must be an equal gain in the sum of the other two.

In Fig. 37, *bcde* represents a filament of water flowing with steady stream-line motion surrounded by other water moving with the same velocity as that of the filament. Under these conditions

the frictional loss that occurs will be extremely small and for the present it will be ignored. Every particle of water passing the section *bc* will, a little later, pass the section *de* and no water will pass the section *de* which did not previously pass *bc*.

Consider now the forces acting on this filament of water. On the section *bc* whose area is a_1 there is a normal pressure in the direction of flow of intensity p_1 producing motion. On the section *de* whose area is a_2 there is a normal intensity of pressure p_2 parallel with the direction of flow and resisting motion. On the lateral surface of the filament, indicated by the lines *bd* and *ce*, there is a system of forces acting normal to the direction of motion, which have no effect on the flow and can therefore be neglected. The force of gravity, equal to the weight of the filament, acts downward. The work performed on the filament by the three forces will now be investigated.

Consider that in the time dt the particles of water at *bc* move to $b'c'$ with a velocity v_1. In the same time interval the particles at *de* move to $d'e'$ with a velocity v_2. Since there is continuity of flow,

$$a_1v_1dt = a_2v_2dt.$$

The work, G_1, done by the force acting on the section *bc* in the time dt is the product of the total force and the distance through which it acts, or

$$G_1 = p_1a_1v_1dt \text{ foot-pounds.} \qquad (6)$$

Similarly the work done on the section *de* is

$$G_2 = -p_2a_2v_2dt \text{ foot-pounds,} \qquad (7)$$

being negative because p_2 is opposite in sense to p_1 and resists motion.

The work done by gravity on the entire mass in moving from the position *bcde* to $b'c'd'e'$ is the same as though $bcb'c'$ were moved to the position $ded'e'$ and the mass $b'c'de$ were left undisturbed. The force of gravity acting on the mass $bcb'c'$ is equal to the volume a_1v_1dt times the unit weight w. If z_1 and z_2 represent, respectively, the elevations of the centers of gravity of $bcb'c'$ and $ded'e'$ above the datum plane MN, the distance through which the force of

gravity would act on the mass $bcb'c'$ in moving it to the position $ded'e'$ is $z_1 - z_2$ and the work done by gravity is

$$G_3 = wa_1v_1dt(z_1 - z_2) \text{ foot-pounds.} \quad . \quad . \quad . \quad . \quad (8)$$

The resultant gain in kinetic energy is

$$\frac{Mv_2^2}{2} - \frac{Mv_1^2}{2} = \frac{wa_1v_1dt}{2g}(v_2^2 - v_1^2). \quad . \quad . \quad . \quad . \quad . \quad (9)$$

From fundamental principles of mechanics, the total amount of work done on any mass by any number of forces is equal to the resultant gain in kinetic energy. Therefore from equations (6), (7), (8) and (9).

$$p_1a_1v_1dt - p_2a_2v_2dt + wa_1v_1dt(z_1 - z_2) = \frac{wa_1v_1dt}{2g}(v_2^2 - v_1^2). \quad . \quad . \quad . \quad (10)$$

Dividing through by wa_1v_1dt and transferring, and remembering that $a_1v_1 = a_2v_2$, there results

$$\frac{v_1^2}{2g} + \frac{p_1}{w} + z_1 = \frac{v_2^2}{2g} + \frac{p_2}{w} + z_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

This is known as Bernoulli's equation. It is the mathematical expression of *Bernoulli's theorem* which is in reality the law of conservation of energy applied to flowing water. It may be stated as follows:

Neglecting friction, the total head, or the total amount of energy per pound of water, is the same at every point in the path of flow.

Water invariably suffers a loss of energy through friction in flowing from one point to another. If the direction of flow is from point 1 to point 2, the total energy at 2 must be less than at 1. In order to make equation (11) balance, a quantity, h_f, equal to the loss of energy, or what is equivalent, the loss of head due to friction between the two points, must be added to the right-hand side of the equation. Including the loss of head due to friction Bernoulli's equation becomes

$$\frac{v_1^2}{2g} + \frac{p_1}{w} + z_1 = \frac{v_2^2}{2g} + \frac{p_2}{w} + z_2 + h_f \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

This equation is the basis of all rational formulas used in hydrokinetics. It is the foundation of the science.

45. Application of Bernoulli's Theorem to Hydrostatics.—Although in hydrostatics it is not necessary to make use of Bernoulli's equation, it is interesting to note that it applies to water at rest as well as to water in motion. The points 1 and 2 (Fig. 38) are h_1 and h_2, respectively, below the free surface of a liquid at rest and z_1 and z_2 above the horizontal plane MN. The liquid being at rest, v_1 and v_2 equal zero, and, since without velocity there can be no friction, h_f is zero. Therefore equation (12) reduces to

$$\frac{p_1}{w}+z_1=\frac{p_2}{w}+z_2, \qquad (13)$$

or, transposing,

$$\frac{p_1}{w}-\frac{p_2}{w}=z_2-z_1. \qquad (14)$$

From the figure

$$z_2-z_1=h_1-h_2.$$

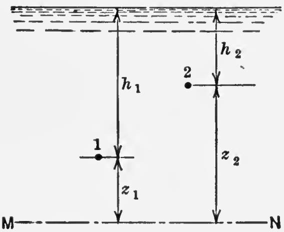

Fig. 38.

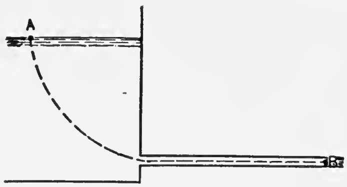

Fig. 39.

Substituting this value of z_2-z_1, equation (14) may be written

$$p_1-p_2=w(h_1-h_2), \qquad (15)$$

which is the same as equation (3), page 11.

46. Bernoulli's Theorem in Practice.—Bernoulli's theorem (Art. 44) is based upon the assumptions of steady flow, stream-line motion and continuity of discharge. Under ordinary conditions water flows with turbulent motion (Art. 90) whereas stream-line motion is assumed in applying Bernoulli's theorem. The effect of turbulence is to increase the losses and therefore this additional loss is included with the loss of head due to friction and no further correction is necessary.

It is permissible to write Bernoulli's equation between any two points on any assumed line of flow provided it is known that there is flow between the two points. Thus, in Fig. 39, which

represents water discharging from a reservoir through a pipe, Bernoulli's equation may be written between points A and B. The relation obtained from this equation is then assumed to hold for each of the filaments in the pipe.

47. Venturi Meter.—An illustration of the practical use of Bernoulli's equation is provided by the Venturi meter. This instrument, which is used for measuring the discharge through pipes, was invented by Clemens Herschel and named by him in honor of the original discoverer of the principle involved. A Venturi meter set in an inclined position is illustrated in Fig. 40.

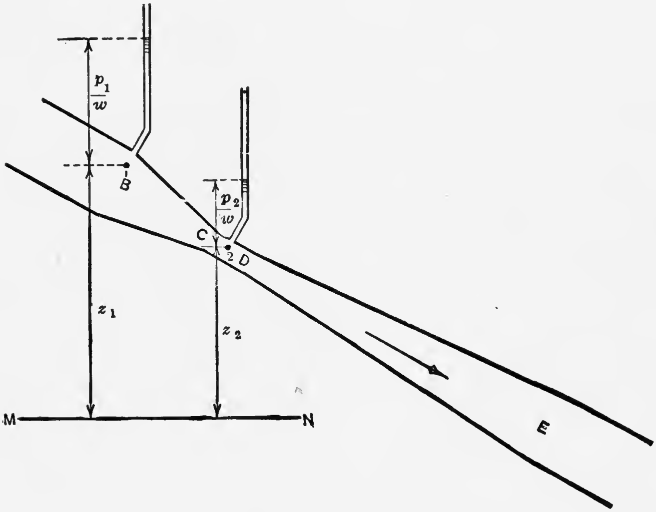

FIG. 40.—Venturi meter.

It consists of a converging section of pipe BC and a longer diverging section DE, the smaller ends being connected by a cylindrical section CD, called the throat. The larger ends B and E, termed the inlet and outlet, respectively, have the same diameter as the pipe line in which the meter is to be installed.

Let a_1, v_1, p_1 and z_1 represent the area, velocity, pressure and elevation, respectively, at point 1 in the inlet. Also let a_2, v_2, p_2 and z_2 represent the corresponding quantities at point 2 in the throat. Writing Bernoulli's equation between points 1 and 2, neglecting friction,

$$\frac{v_1^2}{2g}+\frac{p_1}{w}+z_1=\frac{v_2^2}{2g}+\frac{p_2}{w}+z_2 \quad . \quad . \quad . \quad . \quad . \quad (16)$$

If piezometer tubes are connected at the inlet and throat, the heights of water in these tubes afford a measure of the pressure energy at the two points. Since the relative elevations of points 1 and 2 are also known, the only unknowns in equation (16) are v_1 and v_2. For any given diameters of throat and inlet the corresponding areas can be found, and since

$$Q = a_1v_1 = a_2v_2,$$

v_2 can be expressed in terms of v_1, and substituting this equivalent value in equation (16) leaves v_1 as the only unknown. Solving the equation for v_1 and multiplying the result by a_1 gives the discharge through the pipe.

For a given discharge the difference between elevations of water surfaces in the two piezometers will be the same regardless of

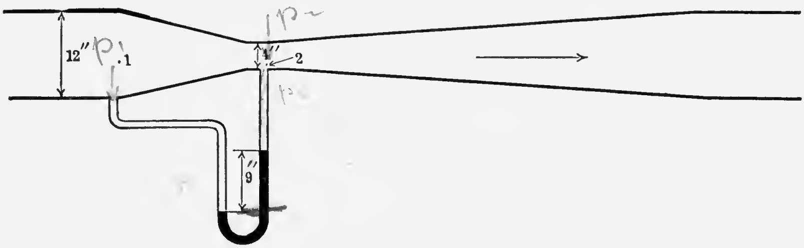

Fig. 41.

whether the meter is horizontal or inclined. If the meter is assumed to be rotated in a vertical plane about point 2 until it is in a horizontal position, the rate of discharge through the meter being unchanged, the total amount of energy contained in the water at the inlet must be the same as before the meter was rotated. Since v_1 has remained constant $\frac{p_1}{w}+z_1$ must also have remained constant. The same reasoning applies to point 2, and hence the difference in elevation of the water levels in the two piezometers is constant for all angles of inclination.

Venturi meters are usually installed in an approximately horizontal position.

Example.—A Venturi meter having a throat 4 in. in diameter is installed in a 12-in. pipe line. A mercury U-tube connected as shown in Fig. 41 shows a difference in height of mercury columns

of 9 in., the remainder of the tube being filled with water. Find the rate of discharge, Q, in cubic feet per second, neglecting friction.

Writing Bernoulli's equation between points 1 and 2

$$\frac{v_1^2}{2g}+\frac{p_1}{w}+z_1=\frac{v_2^2}{2g}+\frac{p_2}{w}+z_2. \quad \ldots \ldots \quad (16)$$

Since the angle of inclination will not affect the result, the meter can be assumed to be horizontal. Then z_1 and z_2 cancel and equation (16) becomes

$$\frac{v_2^2}{2g}-\frac{v_1^2}{2g}=\frac{p_1}{w}-\frac{p_2}{w}. \quad \ldots \ldots \quad (17)$$

From the given data

$$\frac{p_1}{w}-\frac{p_2}{w}=\frac{9}{12}\times 13.6-\frac{9}{12}=9.45.$$

The areas of circles being proportional to the squares of their diameters,

$$\frac{a_1}{a_2}=\frac{12^2}{4^2}=9,$$

and since the flow is continuous

$$Q=a_1v_1=a_2v_2 \quad \text{and} \quad v_2=\frac{a_1}{a^2}v_1=9v_1.$$

Substituting these results in equation (17) and reducing

$$\frac{80v_1^2}{2g}=9.45$$

$$v_1^2=\frac{9.45\times 64.32}{80}=7.6$$

$$v_1=2.76 \text{ ft. per second.}$$

$$Q=a_1v_1=\frac{1^2\times\pi}{4}\times 2.76=2.17 \text{ cu. ft. per second.}$$

The pressure in the throat of a Venturi meter is always much less than at the entrance. This may be seen from the following equation for a horizontal meter:

$$\frac{v_1^2}{2g}+\frac{p_1}{w}=\frac{v_2^2}{2g}+\frac{p_2}{w} \quad \ldots \ldots \quad (18)$$

Since the velocity increases from point 1 to point 2 there must be a corresponding decrease in pressure, otherwise there would be an increase in the total amount of energy per pound of water.

In the practical use of the Venturi meter the loss of head due to friction, though small, should not be neglected.

Consider first the theoretical equation for the horizontal meter:

$$\frac{v_1^2}{2g}+\frac{p_1}{w}=\frac{v_2^2}{2g}+\frac{p_2}{w} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

or

$$\frac{v_2^2-v_1^2}{2g}=\frac{p_1}{w}-\frac{p_2}{w}=h, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

h being the difference in pressure at points 1 and 2, measured in feet of water column. Since

$$v_2=\frac{a_1}{a_2}v_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

substituting, (20) becomes

$$\frac{\left(\frac{a_1}{a_2}\right)^2 v_1^2-v_1^2}{2g}=h. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

This expression reduces to

$$v_1=\sqrt{\frac{2gh}{\left(\frac{a_1}{a_2}\right)^2-1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Since equation (23) does not include the loss of energy (or head) resulting from friction, it gives a greater velocity than is ever obtained. In order to correct the formula for friction loss an empirical coefficient, K, is applied to it. The discharge through the meter is given by the formula

$$Q=a_1v_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Substituting the value of v_1 given in equation (23) and including the coefficient K, the formula for discharge through a Venturi meter becomes

$$Q=Ka_1\sqrt{\frac{2gh}{\left(\frac{a_1}{a_2}\right)^2-1}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

The value of K is affected by the design of the meter and also by the degree of roughness of its inner surface. It has been found from experiments that K usually lies between 0.97 and 0.99.

48. Pitot Tube.—Fig. 42 illustrates several tubes immersed vertically in a stream of flowing water. The upper ends of the tubes are open and exposed to the atmosphere. At the same depth, h_d, there is an opening in each tube which allows free communication between the tube and water in the stream. The velocity of the water at depth h_d is v.

Tubes (a), (b) and (c) are similar, being bent through an angle of 90°, the tip of each tube being open. When the open end of such a tube is directed against the current as shown by (a), the velocity of the water causes water to rise in the tube a distance

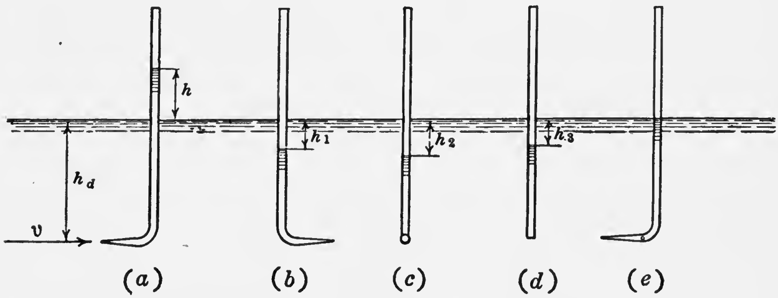

FIG. 42.

h above the free surface of the stream. It is shown later in this article that h is equal to the head due to the velocity v.

If the same tube is placed with the open end directed downstream as in (b) the pressure at the opening is less than h_d and the water surface in the tube is a certain distance, h_1, below the surface of the stream. A similar condition exists when the tube is placed with its lower leg transverse to the stream as shown in (c). In this case, according to experiments by Darcy, the vertical distance, h_2, of the water surface in the tube below the water surface of the stream is a little greater than h_1. Similarly for (d), which is a straight tube open at each end, there is a depression, h_3, of the water column in the tube. The tube (e) is the same as (a) except that the tip of the tube is closed and there is a small hole on each side of the lower leg. If this tube is held with the

lower leg parallel to the direction of flow the water surface in the tube remains at about the same elevation as the water surface of the stream. h_1, h_2 and h_3 are all less than the velocity head, but directly proportional to it. From experiments by Darcy the following approximate values of h_1 and h_2 were obtained:

$$h_1 = 0.43\frac{v^2}{2g} \quad \text{and} \quad h_2 = 0.68\frac{v^2}{2g}.$$

The conditions of flow affecting the height of water column in tube (d) are similar to those encountered when piezometer tubes (Art. 21) extend through the conduit walls into the stream. Piezometer tubes are designed to measure pressure head only and in order that their readings may be affected a minimum amount

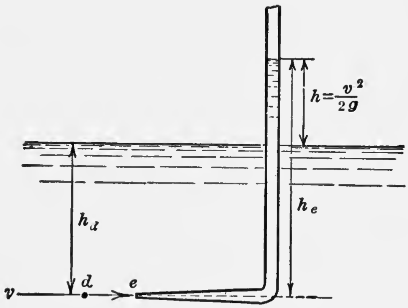

FIG. 43.—Pitot tube.

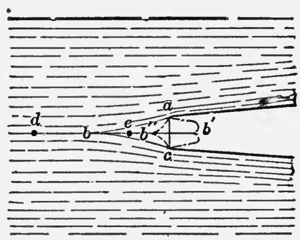

FIG. 44.

by the movement of the water, their ends should be set flush with the inner surface of the conduit and they should never project beyond this surface.

A bent L-shaped tube with both ends open, similar to Fig. 42 (a), is called a *Pitot tube*, from the name of the French investigator who first used such a device for measuring the velocity of flowing water.

When a Pitot tube is first placed in the position shown in Fig. 43, water enters the opening at e until the water surface within the tube rises a distance h above the surface of the stream. A condition of equilibrium is then established and the quantity of water contained within the tube will remain unchanged as long as the flow remains steady. Assuming stream-line motion, the conditions of flow near the entrance to the tube will be as shown in Fig. 44. There will be a volume of dead water in the tube, the

upstream limit of which is not definitely known, but which may be represented by some line such as abc or $ab'c$ or by the intermediate line $ab''c$. Since stream-line motion is assumed, there must be some such surface of quiescent water on the adjacent upstream side of which particles of water will be moving with an extremely low velocity.

Consider a particle of water flowing from d to e, d being on the axis of the tube far enough upstream so that the velocity is not affected by the presence of the tube, e being on the upstream surface of the quiescent water above referred to and at the same elevation as d. As this particle flows from d to e its velocity is gradually retarded from v_d to practically zero at e. The velocity head at e may therefore be called zero.

Based on the above assumptions and neglecting friction, Bernoulli's equation between the points d and e may be written

$$\frac{v_d^2}{2g}+\frac{p_d}{w}+0=0+\frac{p_e}{w}+0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

From Fig. 43

$$\frac{p_e}{w}=h_e \quad \text{and} \quad \frac{p_d}{w}=h_d,$$

since from the figure $h_e-h_d=h$

$$\frac{p_e}{w}-\frac{p_d}{w}=h.$$

Substituting in equation (26)

$$\frac{v_d^2}{2g}=h. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Since d is a point at any depth, the general expression may be written

$$h=\frac{v^2}{2g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

or

$$v=\sqrt{2gh}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Hence the velocity head at d is transformed into pressure head at e and because of this increased pressure inside the tube a column of water will be maintained whose height is $\frac{v^2}{2g}$ above the water level outside.

Pitot tubes of the type shown in Fig. 43 are not practicable for measuring velocities because of the difficulty of determining the height of the water surface in the tube above the surface of the stream. In order to overcome this difficulty Darcy used an instrument with two L-shaped tubes as shown in Fig. 45. One tube is directed upstream and the other downstream, the two tubes being joined at their upper ends to a single tube connected with an air pump and provided with a stopcock at A. By opening the stopcock and pumping some of the air from the tubes both water columns are raised an equal amount, since the pressure on their surfaces is reduced equally.

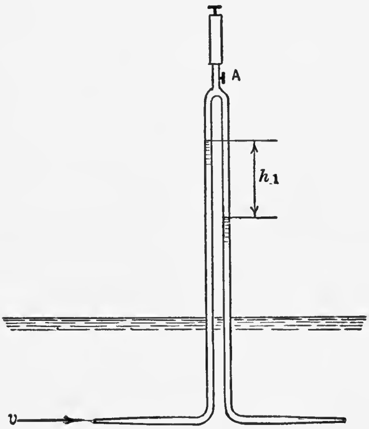

FIG. 45.

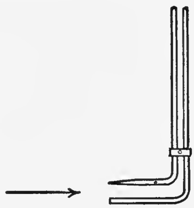

FIG. 46.

The stopcock can then be closed and the difference in height of water columns can be read.

A tube of this kind requires rating, since the downstream leg, which is the same as Fig. 42 (b), does not measure the static pressure of the water. The difference in height of water columns, h_1, is greater than the velocity head, $\frac{v^2}{2g}$, but the ratio between the two is approximately constant. The value of this ratio must be determined experimentally for each instrument.

The tubes shown in Fig. 46 give a difference in elevation of water columns practically equal to the velocity head. The leg measuring static head is similar to Fig. 42 (e).

Pitot tubes may be rated by determining their readings when placed in water flowing with a known velocity or when they are moved at a known rate through still water.

If the Pitot tube is used for measuring velocity in a closed

conduit flowing under pressure, two tubes are absolutely essential, one measuring both pressure and velocity and the other measuring pressure only.

PROBLEMS

1. The diameter of a pipe changes gradually from 6 in. at A to 18 in. at B. A is 15 ft. lower than B. If the pressure at A is 10 lbs. per square inch and at B, 7 lbs. per square inch when there are 5.0 cu. ft. per second flowing, determine:

(*a*) The direction of flow.
(*b*) The frictional loss between the two points.

2. If in Problem 1 the direction of flow is reversed, determine the pressure at A if all other factors, including the frictional loss, remain the same.

3. In Problem 1, determine the diameter of pipe at B in order that the pressure at that point will also be 10 lbs. per square inch, all other factors remaining constant.

4. Determine the discharge in Problem 1, assuming no frictional loss, all other conditions remaining as stated.

5. What would be the difference in pressure in pounds per square inch between A and B, Problem 1, if there were 6.2 cu. ft. per second flowing, neglecting friction.

6. A siphon having a diameter of 6 in. throughout, discharges from a reservoir, A, into the air at B. The summit of the siphon is 6 ft. above the water surface in A and 20 ft. above B. If there is a loss of 3 ft. head between A and the summit and 2 ft. between the summit and B, what is the absolute pressure at the summit in pounds per square inch? Also determine the rate of discharge in cubic feet per second and in gallons per day.

7. In Problem 6 determine the absolute pressure in pounds per square inch at the summit and the discharge in cubic feet per second if the diameter at the summit is 5 in. and at the outlet, 6 in., all other data remaining the same.

8. A flaring tube discharges water from a reservoir at a depth of 36 ft. below the water surface. The diameter gradually increases from 6 in. at the throat to 9 in. at the outlet. Neglecting friction determine the maximum possible rate of discharge in cubic feet per second through this tube. What is the corresponding pressure at the throat?

9. In Problem 8 determine the maximum possible diameter at the outlet at which the tube will flow full.

10. A jet of water is directed vertically upward. At A its diameter is 3 in. and its velocity is 30 ft. per second. Neglecting air friction, determine its diameter at a point 10 ft. above A.

11. Water is delivered by a scoop from a track tank to a locomotive tender that has a speed of 20 mi. per hour. If the entrance to the tender is 7 ft. above the level of the track tank and 3 ft. of head is lost in friction at what velocity will the water enter the tender?

12. In Problem 11 what is the lowest possible speed of the train at which water will be delivered to the tender?

13. A Venturi meter having a diameter of 6 in. at the throat is installed in an 18 in. water main. In a differential gage partly filled with mercury (the remainder of the tube being filled with water) and connected with the meter at the throat and inlet, the mercury column stands 15 in. higher in one leg than in the other. Neglecting friction, what is the discharge through the meter in cubic feet per second?

14. In Problem 13 what is the discharge if there is 1 ft. of head lost between the inlet and the throat, all other conditions remaining the same.

15. In Problem 13 what would be the difference in the level of the mercury columns if the discharge is 5.0 cu. ft. per second and there is 1 ft. of head lost between the inlet and throat?

16. A Venturi meter is installed in a 12-in. water main. If the gage pressure at the meter inlet is 8 lbs. per square inch when the discharge is 3.0 cu. ft. per second determine the diameter of the throat if the pressure at that point is atmospheric. Neglect friction.

17. A flaring tube discharges water from a vessel at a point 10 ft. below the surface on which the gage pressure is 8.5 lbs. per square inch. If the diameter of the throat is 4 in., at which point the absolute pressure is 10 lbs. per square inch, determine the discharge in cubic feet per second, neglecting friction.

18. In Problem 17 what is the diameter of the tube at the discharge end?

CHAPTER VII

FLOW OF WATER THROUGH ORIFICES AND TUBES

49. Description and Definitions.—As commonly understood in hydraulics, an orifice is an opening with a closed perimeter and of regular form through which water flows. If the perimeter is not closed or if the opening flows only partially full the orifice becomes a weir (Art. 68). An orifice with prolonged sides, such as a piece of pipe two or three diameters in length set in the side of a reservoir, is called a *tube*. An orifice in a thick wall has the hydraulic properties of a tube. Orifices may be *circular, square, rectangular* or of any other regular shape.

The stream of water which issues from an orifice is termed the *jet*. An orifice with a sharp upstream edge so formed that water in passing touches only this edge is called a *sharp-edged orifice.* The term *velocity of approach* as applied to orifices means the mean velocity of the water in a channel leading up to an orifice. The portion of the channel where the velocity of approach is considered to occur is designated the *channel of approach.* An orifice is spoken of as a *vertical* or *horizontal* orifice depending upon whether it lies in a vertical or horizontal plane.

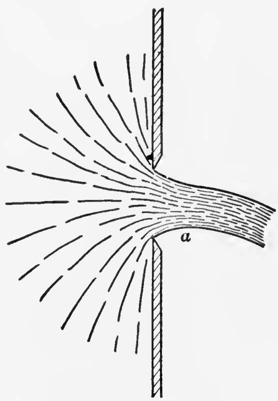

Fig. 47.—Sharp-edged orifice.

50. Characteristics of the Jet.—Fig. 47 represents a sharp-edged, circular orifice. The water particles approach the orifice in converging directions as shown by the paths in the figure. Because of the inertia of those particles whose velocities have components parallel to the plane of the orifice, it is not possible to make abrupt changes in their directions the instant they leave the orifice and they therefore continue to move in curvilinear

paths, thus causing the jet to contract for some distance beyond the orifice. This phenomenon of contraction is referred to as the *contraction of the jet* and the section where contraction ceases is called the *vena contracta.* The vena contracta has been found to be at a distance equal to about one-half the diameter of the orifice from the plane of the orifice, at a in figure.

Beyond the vena contracta the cross-sectional area of the jet does not undergo any change excepting insofar as it is affected by gravity. If the direction of the jet is vertically upward (Fig. 51), or if it has an upward component, the force of gravity retards its velocity and thus increases its cross-sectional area; and, conversely, if the direction of the jet has a downward component, gravity increases its velocity and decreases its cross-sectional area.

If a jet discharges into the air the pressure within the jet at its vena contracta and beyond is atmospheric pressure. This may be seen by investigating the conditions which will result from assuming pressures greater or less than atmospheric pressure. If, for example, the pressure within a cross-section is assumed to be greater than atmospheric pressure, there will be an unbalanced pressure along every radius of the section—that is, the pressure at the center will be greater than at the circumference. Since water is incapable of resisting tensile stress this would cause the jet to expand. In a similar manner if the internal pressure is assumed to be less than atmospheric, since water unconfined is incapable of resisting a compressive force, the unbalanced pressure will produce an acceleration and therefore a further contraction. Since neither expansion nor contraction occurs, it follows that the pressure throughout the vena contracta must be atmospheric pressure.

Between the plane of the orifice and the vena contracta the pressure within the jet is greater than atmospheric pressure because of the centripetal force necessary to change the direction of motion of the particles. That this pressure must be greater than atmospheric can easily be proved by writing Bernoulli's equation between a water particle in the jet back of the vena contracta and another particle in the vena contracta.

The form assumed by jets issuing from orifices of different shapes presents an interesting phenomenon. The cross-section of the jet is similar to the shape of the orifice until the vena con-

tracta is reached. Fig. 48 shows various cross-sections of jets issuing from square, triangular and elliptical orifices. The left-hand diagram in each case is a cross-section of the jet near the vena contracta. The following diagrams are cross-sections at successively greater distances from the orifice. This phenomenon, which is common to all shapes of orifices excepting circular orifices, is known as the inversion of the jet. After passing through the fourth stage shown in the figure the jet reverts to its original form and continues to pass through the cycle of changes described above as long as it flows freely or is not broken up by wind or air friction.

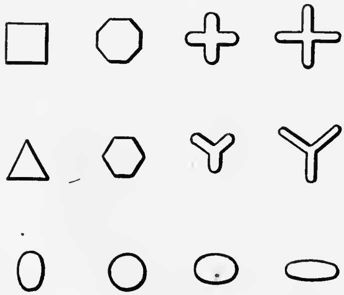

Fig. 48.—Form of jet from square, triangular and elliptical orifices.

51. Fundamental Orifice Formula.—Fig. 49 represents the general case of water discharging through an orifice. In the

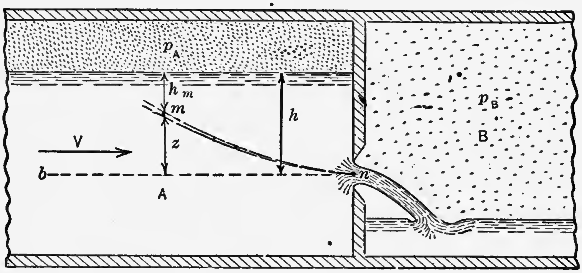

Fig. 49.—Discharge from orifice.

derivation of the fundamental formula it will be assumed that the water flows without friction and also that there is no contraction of the jet and therefore no pressure within the jet in the plane of the orifice. In order to write a general expression applicable to all filaments, it will be necessary to make the further assumption that all of the water particles in a cross-section of the channel of approach flow with the same velocity.

There are two chambers, A and B, the gas pressure in chamber A being p_A and in chamber B being p_B, the relation of p_A to p_B

being such that water will flow from chamber A to chamber B. The flowing water may be considered to be made up of filaments of which mn is one, m being a point in the water in chamber A and n a point in the jet in the plane of the orifice. The filament passes through the orifice at a distance h below the surface of the water. The point m is at a distance h_m below the water surface and at a distance z above n. v_m is the velocity at m and v_n is the velocity at n. Bernoulli's equation may be written between the points m and n as follows:

$$\frac{v_m^2}{2g}+\left(h_m+\frac{p_A}{w}\right)+z=\frac{v_n^2}{2g}+\frac{p_B}{w}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and since $h_m+z=h$

$$h=\frac{v_n^2}{2g}-\frac{v_m^2}{2g}+\frac{p_B}{w}-\frac{p_A}{w} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and

$$v_n=\sqrt{2g\left(h+\frac{v_m^2}{2g}+\frac{p_A-p_B}{w}\right)}. \quad . \quad . \quad . \quad . \quad (3)$$

These formulas are general expressions of relation between velocity and head for any filament.

Since the filaments at different elevations discharge through a vertical orifice under different heads their velocities are not the same. Where, however, the head is large in comparison with the height of the opening, the mean velocity of the jet may be taken as the velocity due to the mean head. The theoretical mean velocity thus obtained may be represented by the symbol v_t. Introducing also the assumption that all of the water particles in a cross-section of the channel of approach flow with the same velocity, V; v_n and v_m in formulas (2) and (3) may be replaced, respectively, by v_t and V, which gives

$$h=\frac{v_t^2}{2g}-\frac{V^2}{2g}+\frac{p_B}{w}-\frac{p_A}{w} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and

$$v_t=\sqrt{2g\left(h+\frac{V^2}{2g}+\frac{p_A-p_B}{w}\right)}. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

From the definition (Art. 49) V is the velocity of approach.

The condition most commonly encountered is that illustrated

in Fig. 50, where the surface of the water and the jet are each exposed to the atmosphere. In this case $p_A = p_B =$ atmospheric pressure and formulas (4) and (5) reduce, respectively, to

$$h = \frac{v_t^2}{2g} - \frac{V^2}{2g} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and

$$v_t = \sqrt{2g\left(h + \frac{V^2}{2g}\right)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If the cross-sectional area of the reservoir or channel leading up to the orifice is large in comparison with the area of the orifice the velocity of approach, V, may be called zero and equations (6) and (7) reduce, respectively, to

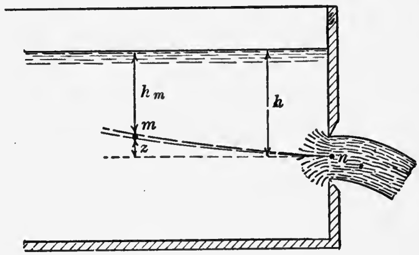

FIG. 50.—Orifice with water surface and jet subjected to equal pressures.

$$h = \frac{v_t^2}{2g} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and

$$v_t = \sqrt{2gh}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

These formulas express the theoretical relation between head and velocity for an orifice discharging from a relatively large body of water whose surface is subjected to the same pressure as the jet. It is under this condition that discharge from orifices ordinarily occurs and the above formulas are the ones most commonly used. Since these formulas also express the relation between potential head and velocity head (Art. 43) they have a wide application in hydraulics.

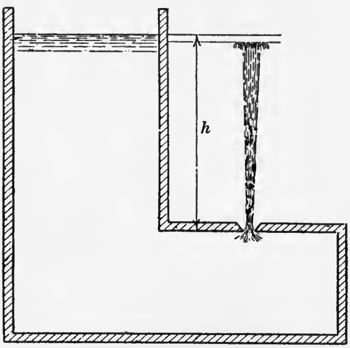

FIG. 51.—Horizontal orifice.

Formula (9) is also the formula for the velocity acquired by a body falling a distance h through space. The theoretical velocity of water flowing through an orifice is therefore the velocity acquired

by a body falling freely *in vacuo* through a distance equal to the head on the orifice. This principle, discovered by Torricelli in 1644, is known as *Torricelli's theorem.* Fig. 51 illustrates a horizontal orifice discharging under a head h. According to Torricelli's theorem the jet should rise to a height h, but experiments show that the actual height to which the jet rises is slightly less than h. The discrepancy is due to the retarding effects of friction and viscosity. This matter is discussed more fully in the following article.

52. Orifice Coefficients.—The assumptions which were made in the derivation of formula (5) may be summarized briefly as follows:

(*a*) All water particles in a cross-section of the channel of approach flow with the same velocity.

(*b*) There is no contraction of the jet.

(*c*) The water flows without friction.

Since these conditions do not in reality exist, it is necessary to modify the derived formulas to make them applicable to actual conditions. To accomplish this, three empirical coefficients are applied to formula (5), there being one coefficient to correct for the difference between the assumed conditions and the actual conditions for each of the above assumptions. The method of correcting for each assumption will be discussed in the order given above.

(*a*) *Correction for non-uniformity of velocity in cross-section of channel of approach.* The effect of the variation in velocity in a cross-section of the channel of approach—that is, the variation in the velocity with which the water particles in the different filaments approach the orifice, is similar to the effect of this condition on the discharge over weirs. The matter being of relatively much greater importance in this connection is taken up under weirs and will not be discussed here. (See Art. 72 (*a*).)

The commonly accepted method of modifying formula (5) so as to have it include this correction is to apply a coefficient α to the term $\frac{V^2}{2g}$. The value of α has not been determined for orifices. It varies with the distribution of velocities in the channel of approach and is always greater than unity. With the coefficient

α included, calling v' the velocity after the correction has been applied, formula (5) becomes

$$v'=\sqrt{2g\left(h+\alpha\frac{V^2}{2g}+\frac{p_A-p_B}{w}\right)}. \quad . \quad . \quad . \quad . \quad (10)$$

(*b*) *Correction for contraction.*—The ratio of the cross-sectional area of the jet at the vena contracta to the area of the orifice is called the *coefficient of contraction.* Thus, if a' and a are, respectively, the cross-sectional area of the jet at the vena contracta and the area of the orifice and C_c is the coefficient of contraction,

$$C_c=\frac{a'}{a} \quad \text{or} \quad a'=C_c a.$$

If v is the actual mean velocity in the vena contracta the discharge through the orifice is

$$Q=a'v=C_c av. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The mean value of C_c is about 0.62. It varies slightly with the head and size of orifice.

(*c*) *Correction for friction.*—The velocity of the jet suffers a retardation due to the combined effects of friction and viscosity. The ratio of the actual mean velocity, v, to the velocity, v', which would exist without friction, has been termed the *coefficient of velocity,* but it might more properly be called the coefficient of friction. Designating the coefficient of velocity by the sumbol C_v

$$C_v=\frac{v}{v'} \quad \text{or} \quad v=C_v v'.$$

The average value of C_v for a sharp-edged orifice is about 0.98.

Substituting the value of v' given in formula (10), the general formula for mean velocity of a jet issuing from an orifice, with the two coefficients to correct, respectively, for friction and the assumption of uniform velocity in a cross-section of the channel of approach, becomes

$$v=C_v\sqrt{2g\left(h+\alpha\frac{V^2}{2g}+\frac{p_A-p_B}{w}\right)}. \quad . \quad . \quad . \quad . \quad (12)$$

If the pressures p_A and p_B are equal

$$v=C_v\sqrt{2g\left(h+\alpha\frac{V^2}{2g}\right)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

If the velocity of approach is so small that it can be called zero without introducing an appreciable error

$$v=C_v\sqrt{2gh}. \qquad (14)$$

Substituting the value of v given by formula (12) in (11) the general formula for discharge through an orifice with the three corrective coefficients becomes

$$Q=C_cC_va\sqrt{2g\left(h+\alpha\frac{V^2}{2g}+\frac{p_A-p_B}{w}\right)}. \qquad (15)$$

It is usual to combine C_cC_v into a single coefficient, C, called the *coefficient of discharge*. Substituting C for C_cC_v, the general formula for discharge is

$$Q=Ca\sqrt{2g\left(h+\alpha\frac{V^2}{2g}+\frac{p_A-p_B}{w}\right)}. \qquad (16)$$

If the pressures p_A and p_B are equal

$$Q=Ca\sqrt{2g\left(h+\alpha\frac{V^2}{2g}\right)}. \qquad (17)$$

If the velocity of approach V is so small that it can be considered to equal zero

$$Q=Ca\sqrt{2gh}. \qquad (18)$$

As orifices are ordinarily used, the pressures p_A and p_B are equal and the velocity of approach is so small that it may be neglected without appreciable error. Formula (18) is therefore recognized as the common discharge formula for an orifice.

In the remainder of this chapter, if not otherwise specified, it will be assumed that the pressure on the water surface is the same as the pressure on the jet, and the velocity of approach will be considered to be so small as to be negligible. Formulas (14) and (18) then become the respective formulas for mean velocity and discharge.

Numerical values of C_c and C_v may be obtained experimentally but an accurate determination is extremely difficult. C_c may be obtained approximately by measuring the diameters of the vena contracta and orifice with calipers, the coefficient of contraction being equal to the ratio of the squares of their respective

Coefficients of Discharge (C) for Circular Orifices

Head h in feet	Diameter of orifice in feet						
	0.02	0.04	0.07	0.1	0.2	0.6	1.0
0.4	.....	0.637	0.624	0.618			
0.6	0.655	.630	.618	.613	0.601	0.593	
0.8	.648	.626	.615	.610	.601	.594	0.590
1.0	.644	.623	.612	.608	.600	.595	.591
1.5	.637	.618	.608	.605	.600	.596	.593
2.0	.632	.614	.607	.604	.599	.597	.595
2.5	.629	.612	.605	.603	.599	.598	.596
3.0	.627	.611	.604	.603	.599	.598	.597
4.0	.623	.609	.603	.602	.599	.597	.596
6.0	.618	.607	.602	.600	.598	.597	.596
8.0	.614	.605	.601	.600	.598	.596	.596
10.0	.611	.603	.599	.598	.597	.596	.595
20.0	.601	.599	.597	.596	.596	.596	.594
50.0	.596	.595	.594	.594	.594	.594	.593
100.0	.593	.592	.592	.592	.592	.592	.592

Coefficients of Discharge (C) for Square Orifices

Head h in feet	Side of square in feet						
	0.02	0.04	0.07	0.1	0.2	0.6	1.0
0.4	.....	0.643	0.628	0.621			
0.6	0.660	.636	.623	.617	0.605	0.598	
0.8	.652	.631	.620	.615	.605	.600	0.597
1.0	.648	.628	.618	.613	.605	.601	.599
1.5	.641	.622	.614	.610	.605	.602	.601
2.0	.637	.619	.612	.608	.605	.604	.602
2.5	.634	.617	.610	.607	.605	.604	.602
3.0	.632	.616	.609	.607	.605	.604	.603
4.0	.628	.614	.608	.606	.605	.603	.602
6.0	.623	.612	.607	.605	.604	.603	.602
8.0	.619	.610	.606	.605	.604	.603	.602
10.0	.616	.608	.605	.604	.603	.602	.601
20.0	.606	.604	.602	.602	.602	.601	.600
50.0	.602	.601	.601	.600	.600	.599	.599
100.0	.599	.598	.598	.598	.598	.598	.598

diameters. A Pitot tube may be used to determine approximately velocities in the vena contracta.

The coefficient of discharge may be obtained with great accuracy by measuring the quantity of water flowing from an orifice of known dimensions in a given time and determining the ratio between this discharge and the theoretical discharge. Since in practice it is usually the discharge from orifices that is required, it is the coefficient of discharge that is of greatest value to engineers. An average value of the coefficient of discharge is about 0.60. It is not a constant, but varies with the head and also with the shape and size of the opening. On page 79 are tables of values of C for circular and square orifices taken from Hamilton Smith's Hydraulics. Sharp-edged orifices provide an accurate means of measuring small rates of discharge.

53. Algebraic Transformation of Formula with Velocity of Approach Correction.—The fundamental orifice formula with velocity of approach correction as derived in Art. 52 is

$$Q=Ca\sqrt{2g\left(h+\alpha\frac{V^2}{2g}\right)}. \qquad (17)$$

By definition $V=Q/A$ where A is the cross-sectional area of the stream in the channel of approach. Substituting this value of V and reducing, equation (17) becomes

$$Q=\frac{Ca\sqrt{2gh}}{\sqrt{1-\alpha C^2\frac{a^2}{A^2}}}. \qquad (19)$$

Expanding the denominator by the binomial theorem gives a diminishing series and dropping all terms excepting the first two since they will be very small quantities, the formula may be expressed in the approximately equivalent form

$$Q=Ca\sqrt{2gh}\left(1+\frac{\alpha C^2}{2}\frac{a^2}{A^2}\right). \qquad (20)$$

The term within the parenthesis is the velocity of approach corrective factor. It becomes unity when the ratio of the orifice to the cross-sectional area of the stream in the channel of approach is so small that it may be considered zero. Where a correction for velocity of approach is required, formula (20) will be found more convenient than formula (17).

54. Head Lost in an Orifice.—Consider water to be discharging from an orifice under a head h (Fig. 52). Because of friction, the velocity of discharge will be less than $\sqrt{2gh}$ or, from formula (14), page 78

$$v = C_v\sqrt{2gh}. \quad \ldots \ldots \ldots \quad (14)$$

The head producing discharge is therefore,

$$h = \frac{1}{C_v^2}\frac{v^2}{2g}. \quad \ldots \quad (19)$$

That is, h is the total head, including the lost head. The

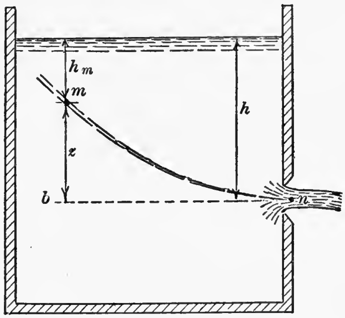

FIG. 52.—Sharp-edged orifice.

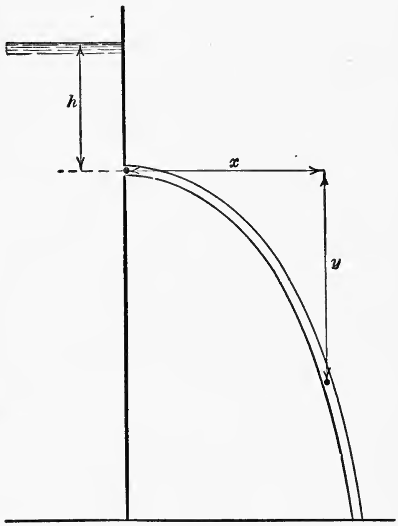

FIG. 53.—Path of jet.

head that is not lost is the velocity head due to the actual velocity v. Therefore,

Lost head = Total head − velocity head or, placing the symbol h_0 for lost head,

$$h_0 = \frac{1}{C_v^2}\frac{v^2}{2g} - \frac{v^2}{2g} = \left(\frac{1}{C_v^2} - 1\right)\frac{v^2}{2g}. \quad \ldots \ldots \quad (22)$$

For a sharp-edged orifice, since $C_v = 0.98$, $h_o = 0.041\,\frac{v^2}{2g}$.

Since $v = C_v v_t$, formula (22) reduces to

$$h_0 = (1 - C_v^2)h, \quad \ldots \ldots \ldots \ldots \quad (23)$$

and substituting the value of C_v for a sharp-edged orifice

$$h_0 = 0.040\ h.$$

Formulas (22) and (23) are fundamental and are applicable to any orifice or tube whose coefficient of velocity is known.

55. Path of Jet.—When water issues from an orifice the direction of the jet is at first normal to the plane of the orifice but, for orifices not in a horizontal plane, the force of gravity causes it immediately to begin to curve downward. Let x (Fig. 53) be the abscissa and y the ordinate of any point in the path of a jet discharging from a vertical orifice. The space x will be described uniformly in a certain time t and if v is the velocity with which water leaves the orifice

$$x=vt.$$

The jet has a downward acceleration which conforms to the law of falling bodies and therefore

$$y=\frac{gt^2}{2}.$$

Eliminating t between the two equations

$$x^2=\frac{2v^2}{g}y, \qquad (24)$$

which is the equation of a parabola with its vertex at the orifice. Since by formula (14), page 78,

$$v=C_v\sqrt{2gh}, \qquad (14)$$

equation (24) may also be written

$$x^2=4C_v^2hy. \qquad (25)$$

This formula indicates an experimental method of obtaining C_v; x, y and h may be measured and substituted in the formula and C_v may be computed

56. Orifices under Low Heads.—Where the head on a vertical orifice is small in comparison with the height of the orifice there is theoretically an appreciable difference between the discharge obtained by assuming the mean velocity to be that due to the mean head and the discharge obtained by taking into consideration the variation in head. The exact formula for rectangular orifices is derived as follows:

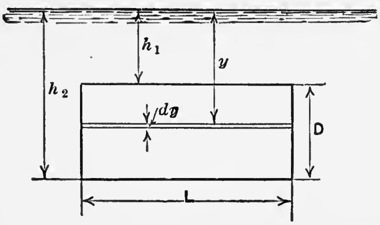

FIG. 54.—Rectangular orifice.

Fig. 54 shows a rectangular orifice of width L and height D,

the water surface and jet being each subjected to atmospheric pressure. h_1 and h_2 are the respective heads on the upper and lower edges of the orifice. Neglecting velocity of approach, the theoretical discharge through any elementary strip of area Ldy at a distance y below the water surface, is given by the equation

$$dQ_t = L\sqrt{2gy}dy$$

which integrated between the limits of h_2 and h_1 gives

$$Q_t = \tfrac{2}{3}L\sqrt{2g}(h_2^{3/2} - h_1^{3/2}). \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

This formula gives the theoretical discharge from an orifice where the pressures on the water surface and jet are equal and the velocity of approach is considered to be zero. When h_1 is zero—that is, when the water surface does not touch the upper edge of the opening, the formula reduces to

$$Q_t = \tfrac{2}{3}\sqrt{2g}Lh_2^{3/2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

which is the theoretical formula for discharge over a weir without velocity of approach correction (see Art. 70).

To make formula (26) applicable to actual conditions a coefficient of discharge must be introduced and the formula becomes

$$Q = \tfrac{2}{3}CL\sqrt{2g}(h_2^{3/2} - h_1^{3/2}). \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Values of C for this formula have not been well determined and it is seldom used in practice. Formula (18), which for rectangular orifices may be written

$$Q = CLD\sqrt{2gh}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

h being the head on the center of the orifice and C the coefficient of discharge for rectangular orifices as given on page 79, may be used satisfactorily even for quite low heads since these values of C include corrections for the approximations contained in the formula.

The theoretical difference between formulas (28) and (29) may be shown as follows: h being the head on the center of the rectangle, $h_2 = h + \tfrac{1}{2}D$ and $h_1 = h - \tfrac{1}{2}D$. Substituting these values in equation (26) and expanding them by the binomial theorem,

$$Q = CLD\sqrt{2gh}\left(1 - \frac{D^2}{96h^2} - \frac{D^4}{2048h^4} - \frac{D^6}{21845h^6} \cdots\right). \quad . \quad (30)$$

This shows that formula (29) always gives a greater discharge than formula (28) if the same value of C is used in each case. For $h=D$, the sum of the infinite series is 0.989 and for $h=2D$, it is 0.997. The theoretical error introduced by using formula (29) is thus about 1 per cent where $h=D$ and 0.3 of 1 per cent where $h=2D$.

In a manner similar to that described above for rectangular orifices, the discharge for a circular orifice may be shown to be

$$Q=\tfrac{1}{4}\pi CD^2\sqrt{2gh}\left(1-\frac{D^2}{128h^2}-\frac{5D^4}{16348h^4},\ldots\right). \quad . \quad . \quad . \quad (31)$$

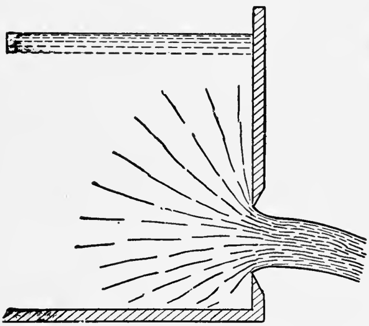

FIG. 55.—Orifice with bottom contraction partially suppressed.

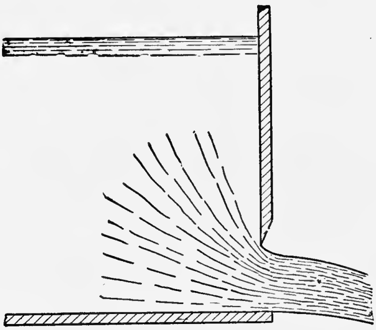

FIG. 56.—Orifice with bottom contraction completely suppressed.

in which D is the diameter of orifice and h is the head on the center of orifice. This formula gives results differing from those obtained by the approximate formula similar to the corresponding formulas for rectangular orifices. If $h=D$ the sum of the series is 0.992 and if $h=2D$ the sum is 0.998. Formula (31) is seldom, if ever, used in practice.

57. Suppression of Contraction.—The effect of constructing an orifice so as to reduce the contraction is to increase the cross-sectional area of the jet and thus to increase the discharge. If an orifice is placed close to a side or the bottom of a reservoir the tendency of the filaments of water to approach the orifice from all directions (Fig. 55) is restricted and some of the filaments must approach in a direction more nearly parallel to the direction of the jet than they would otherwise. If the orifice is flush with one side or the bottom (Fig. 56) the contraction on that side of the orifice will be wholly suppressed.

In a similar manner, rounding the inner edge of the orifice (Fig. 57) reduces contraction. An orifice constructed to conform to the shape of the jet which issues from a sharp-edged orifice

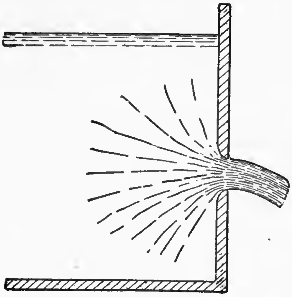

FIG. 57.—Orifice with rounded entrance.

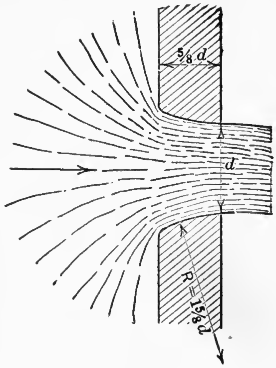

FIG. 58.—Bell-mouth orifice.

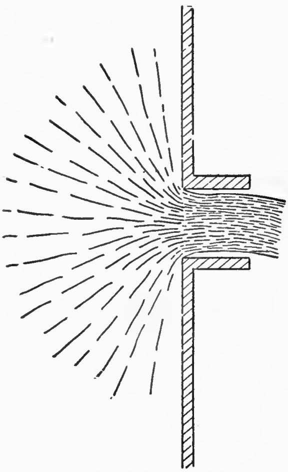

FIG. 59.—Sharp-edged orifice with extended sides.

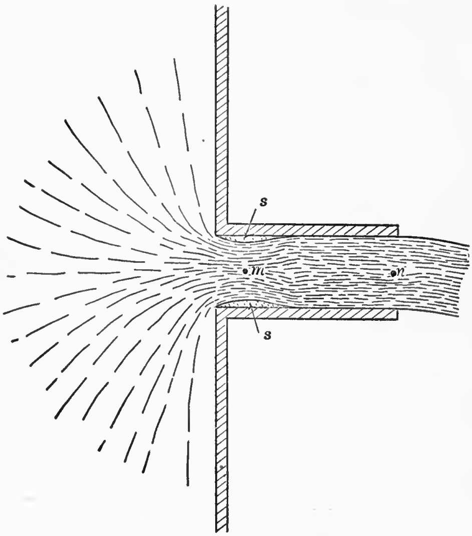

FIG. 60.—Standard short tube.

(Fig. 58) is called a bell-mouth orifice. The coefficient of contraction of such an orifice approaches very close to unity.

58. Standard Short Tube.—Extending the sides of an orifice does not affect the discharge so long as the jet springs clear. The

orifice illustrated in Fig. 59 has a sharp upstream corner and the conditions of flow are the same as for a sharp-edged orifice in a thin plate.

When the jet touches the sides of the orifice the conditions of flow are changed. A circular orifice with a sharp edge having sides extended to about $2\frac{1}{2}$ diameters is called a *standard short tube.* The jet is contracted by the edge of the orifice, as at m (Fig. 60), and for low heads it will expand and fill the tube. For high heads the jet may at first spring clear of the sides of the tube, but by temporarily stopping the tube at its discharging end and allowing the water to escape, the tube can be made to flow full. The moving water carries with it a portion of the air which is entrapped in the space, s, causing a pressure less than atmospheric pressure. The result is to increase the head under which water enters the orifice and therefore the discharge is greater than occurs from a sharp-edged orifice of the same diameter discharging freely into the air.

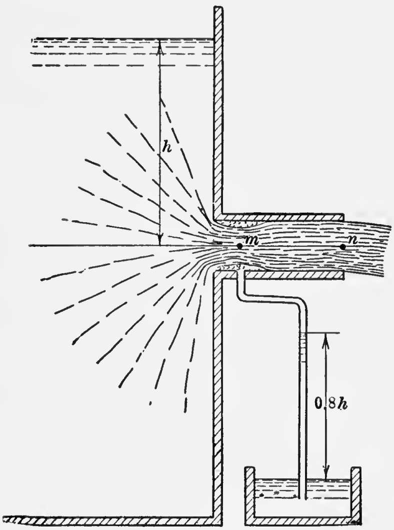

Fig. 61.—Standard short tube.

Conditions at the outlet end of the tube will first be considered. By writing Bernoulli's equation between a point in the reservoir where the velocity of approach may be considered zero and a point, n, in the outlet (Fig. 61) there is obtained, as for an orifice (Art. 51) the relation

$$v_t = \sqrt{2gh}. \quad . \quad . \quad (9)$$

Since the tube flows full, the coefficient of contraction at the outlet equals unity. It has been found experimentally that the coefficient of discharge, C, and therefore the coefficient of velocity, C_v, for the outlet equals approximately 0.82, the value of the coefficient varying slightly with the head and diameter of tube. Therefore from formula (14), page 78

$$v = C_v\sqrt{2gh} = 0.82\sqrt{2gh}, \quad . \quad . \quad . \quad . \quad . \quad (32),$$

and since the coefficient of contraction is unity,

$$Q = Ca\sqrt{2gh} = 0.82a\sqrt{2gh}. \qquad (33)$$

The discharge is thus about one-third greater than for a sharp-edged orifice of the same diameter.

To investigate conditions at the contracted portion of the jet, Bernoulli's equation may be written between a point on the water surface, where the velocity is considered zero and the pressure is atmospheric, and a point m in the contracted portion of the jet. Thus

$$0 + 34 + h = \frac{v_m^2}{2g} + \frac{p_m}{w} + \text{lost head}. \qquad (34)$$

Assuming the coefficient of contraction at m to be 0.62, the same as for a sharp-edged orifice discharging freely into the air, and writing the equation of continuity between m and n

$$v_m \times 0.62a = v \times a$$

or

$$v_m = 1.61v. \qquad (35)$$

The head lost between the reservoir and m (page 81) is $0.04\frac{v_m^2}{2g}$. Substituting these values, equation (34) becomes

$$0 + 34 + h = \frac{(1.61v)^2}{2g} + \frac{p_m}{w} + 0.04\frac{(1.61v)^2}{2g}, \qquad (36)$$

and substituting v from equation (32) and reducing,

$$\frac{p_m}{w} = 34 - 0.8h. \qquad (37)$$

There exists, therefore, a partial vacuum at m which will lift a water column $0.8h$ (Fig. 61), the pressure being $0.8wh$ less than atmospheric pressure. This has been confirmed experimentally. Evidently the relation does not hold when $0.8h$ becomes greater than 34 ft., or when the head becomes greater than approximately 42.5 ft., since this condition gives a negative value to $\frac{p_m}{w}$ in equation (37) which is not possible.

The *lost head* in the entire length of a standard short tube (see Art. 54) is given by the formula

$$h_0=\left(\frac{1}{C_v^2}-1\right)\frac{v^2}{2g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

and since $C_v=0.82$, the formula gives $h_0=0.50\,\frac{v^2}{2g}$.

This case is important since the entrance to a pipe, where the end of the pipe is flush with a vertical wall, is usually considered as a standard short tube and the head lost at entrance to the pipe is taken as the head lost in a standard short tube (see page 156).

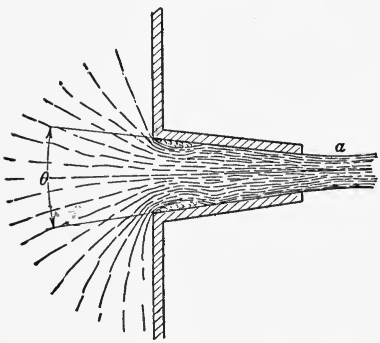

FIG. 62.—Converging tube with sharp-cornered entrance.

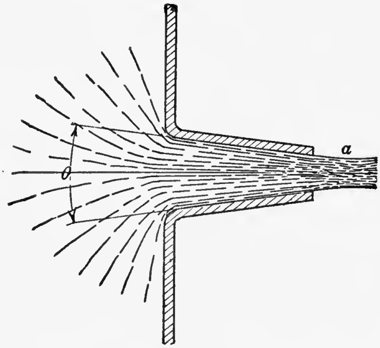

FIG. 63.—Converging tube with rounded entrance.

59. Converging Tubes.—Converging tubes having a circular cross-section are frustums of cones with the larger end adjacent to the reservoir. They may have a sharp-cornered entrance as in Fig. 62 or a rounded entrance as in Fig. 63. The jet contracts slightly at a, just beyond the end of the tube. The coefficient of contraction, C_c, decreases as the angle of convergence, θ, increases; becoming 0.62 for $\theta=180°$ when the tube becomes a sharp-edged orifice. The coefficient of velocity, C_v, on the other hand, decreases as θ decreases. As for any orifice

$$Q=C_cC_va\sqrt{2gh}=Ca\sqrt{2gh}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

The following table gives coefficients for converging, conical tubes with sharp-cornered entrances, interpolated from experiments by d'Aubuisson and Castel. These results are interesting in that they show the general laws of variation of coefficients but, on

account of the small models used in the experiments, they should not be taken as generally applicable to all tubes of this type.

COEFFICIENTS FOR CONICAL CONVERGING TUBES

Coefficient	Angle of convergence, θ (Fig. 62)								
	0°	5°	10°	15°	20°	25°	30°	40°	50°
C_v	0.829	0.911	0.947	0.965	0.971	0.973	0.976	0.981	0.984
C_c	1.000	.999	.992	.972	.952	.935	.918	.888	.859
C	0.829	.910	.939	.938	.924	.911	.896	.871	.845

The coefficient of velocity and therefore the coefficient of discharge is increased by rounding the entrance (Fig. 63), since this reduces the head lost in the tube. The coefficient of contraction will not be materially changed. Exact values of coefficients will depend upon the extent to which the corner is rounded. Maximum discharge is obtained when the shape of the entrance conforms to the shape of the contracted jet.

60. Nozzles.—A nozzle is a converging tube attached to the end of a pipe or hose. The nozzle increases the velocity of the issuing jet, thus increasing the range of distance which it covers. Fig. 64 illustrates two types of nozzles in common use. Each of these has a cylindrical tip of such length that it will flow full, thus preventing contraction and increasing the discharge. The converging part of the tube may be the frustum of a cone as in Fig. 64 (*a*) or the inside may be convex as in (*b*). Each of these shapes gives an efficient stream. The following mean values of coefficients of discharge for smooth nozzles, similar to Fig. 64 (*a*), having a diameter at the base of 1.55 in., have been determined from experiments by Freeman.[1]

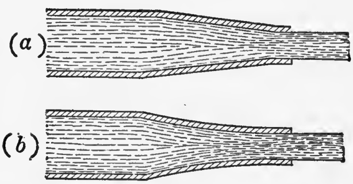

FIG. 64.—Nozzles.

Diameter in inches.....	$\frac{3}{4}$	$\frac{7}{8}$	1	$1\frac{1}{8}$	$1\frac{1}{4}$	$1\frac{3}{8}$
Coefficient of discharge..	0.983	0.982	0.980	0.976	0.971	0.959

[1] JOHN R. FREEMAN: Experiments Relating to Hydraulics of Fire Streams, *Trans.* Amer. Soc. Civ. Eng., vol. 21, pp. 303–482 (1889).

Since the coefficient of contraction is unity, the coefficients of discharge given above are also the coefficients of velocity.

The nozzle being a form of tube to which formula (22), page 81, applies, the head lost in the nozzle is

$$h_0=\left(\frac{1}{C_v^2}-1\right)\frac{v^2}{2g}, \quad \ldots \ldots \ldots \quad (22)$$

or substituting values of C_v given in the above table,

$$h_0=(0.04 \text{ to } 0.09)\frac{v^2}{2g}, \quad \ldots \ldots \quad (39)$$

in which v is the mean velocity at the outlet of the nozzle.

Expressed as a function of the velocity, v_1, in the hose or pipe having a diameter D, the diameter of the nozzle being d,

$$h_0=(0.04 \text{ to } 0.09)\left(\frac{D}{d}\right)^4\frac{v_1^2}{2g}. \quad \ldots \ldots \quad (40)$$

Bernoulli's equation, for a horizontal nozzle, may be written between a point at entrance to the nozzle and a point in the jet as follows:

$$\frac{p_1}{w}+\frac{v_1^2}{2g}=\frac{v^2}{2g}+\text{lost head}, \quad \ldots \ldots \quad (41)$$

in which p_1 is the gage pressure at the entrance, v_1 is the velocity at entrance and v is the velocity in the jet. From this equation the pressure at the base of the nozzle, p_1, may be determined if the discharge is known or the discharge may be determined if p_1 is known.

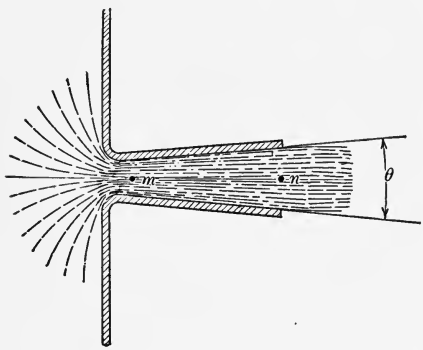

Fig. 65.—Diverging tube.

61. Diverging Tubes.—Fig. 65 represents a conical diverging tube, having rounded entrance corners, so that all changes in velocity occur gradually. Such a tube, provided the angle of flare is not too great nor the tube too long, will flow full. The theoretical velocity, v_t, at the outlet of the tube, obtained in the same manner as for an orifice (Arts. 51 and 52), is

$$v_t=\sqrt{2gh}, \quad \ldots \ldots \ldots \quad (9)$$

the actual mean velocity being

$$v = C_v \sqrt{2gh}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where C_v is the coefficient of velocity at the outlet.

Experiments indicate that even under favorable conditions the value of C_v is small. Venturi and Eytelwein, experimenting with a tube 8 in. long, 1 in. in diameter at the throat and 1.8 in. in diameter at the outer end, obtained results which give a value of C_v of about 0.46. The lost head (formula (22), Art. 54) was, therefore, approximately $0.79h$.

Even with this large loss of head the discharge through the tube was about two and one-half times the discharge from a sharp-edged orifice having the same diameter as the throat of the tube.

The greater portion of the loss of head occurs between the throat and outlet of the tube where the stream is expanding and thus has a tendency to break up in eddies with a waste of energy. Experiments by Venturi indicate that an included angle, θ, of about 5° and a length of tube about nine times its least diameter give the most efficient discharge. A diverging tube, such as that shown in Fig. 65, is commonly called a Venturi tube.

The pressure head at the throat is evidently less than atmospheric pressure. This may be shown by writing Bernoulli's equation between m and n. When the throat is so small that Bernoulli's equation gives a negative absolute pressure at m, formula (14) no longer holds. The conditions are similar to those already described for a standard short tube, Art. 58.

62. Borda's Mouthpiece.—Since the contraction of a jet issuing from an orifice is caused by the water entering the orifice from various directions inclined to the axis of the orifice, it follows that the greater the angle between the extreme directions the greater will be the contraction of the jet. The extreme case occurs in Borda's mouthpiece (Fig. 66), where the water approaches the orifice from all directions. This mouthpiece consists of a thin tube projecting into the reservoir about one diameter. The proportions are such that the jet springs clear of the walls of the tube. Borda's mouthpiece is of interest because it is possible to obtain its coefficient of contraction by rational methods.

The cross-sectional area of the jet at the vena contracta, mn,

is a' and the velocity at this section is v. If a is the area of the opening, the coefficient of contraction $C_c = a'/a$.

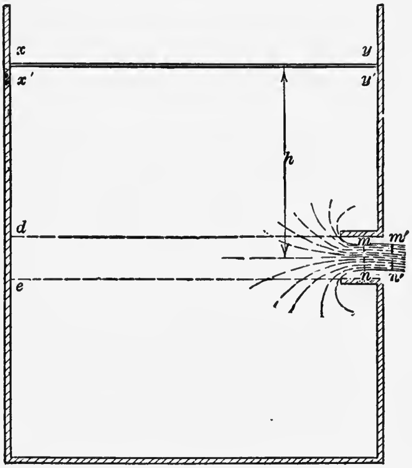

Fig. 66.—Borda's mouthpiece.

The size of the reservoir is assumed to be so large in comparison with the area of the orifice that the velocity of the water within the reservoir may be neglected and that the pressure on the walls will, therefore, follow the laws of hydrostatics. Excepting the pressure acting on the horizontal projection *de* of the mouthpiece on the opposite wall, the horizontal pressures on the walls will balance each other. The total pressure on *de* is *wah*, which is also the resultant horizontal accelerating force acting on the water entering the mouthpiece.

Consider the mass of water *xymn* to move to the position $x'y'$ $m'n'$ in t seconds. The change in the momentum of the mass considered is the difference in the momentum of the mass $xx'yy'$ and $mm'nn'$. But the momentum of $xx'yy'$ is entirely vertical, therefore the change in momentum in a horizontal direction is equal to the momentum of $mm'nn'$, which is produced by the action of the force *wah*.

The mass of $mm'nn'$ is $\dfrac{a'vtw}{g}$ and its momentum is $\dfrac{a'v^2tw}{g}$. The impulse of the force *wah* is *waht*. Equating impulse and change of momentum,

$$waht = \frac{a'v^2tw}{g},$$

therefore,

$$\frac{a'}{a} = \frac{gh}{v^2},$$

and since

$$v = C_v\sqrt{2gh},$$

$$\frac{a'}{a} = C_c = \frac{1}{2C_v^2}.$$

Therefore, assuming the coefficient of velocity to be unity, the coefficient of contraction is theoretically 0.5, or calling the coefficient of velocity 0.98, the same as for a sharp-edged orifice, the coefficient of contraction is 0.52. This value has been verified approximately by experiments.

63. Re-entrant Tubes.—Tubes, having their ends project into a reservoir (Fig. 67), and having a length of about $2\frac{1}{2}$ diameters, are called re-entrant or inward-projecting tubes. The action of water in such tubes is similar to that in standard short tubes (Art. 58), except that the contraction of the jet near the entrance is greater. At the discharge end the tube flows full and the

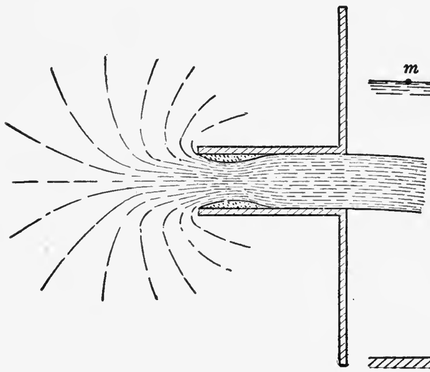

FIG. 67.—Re-entrant tube.

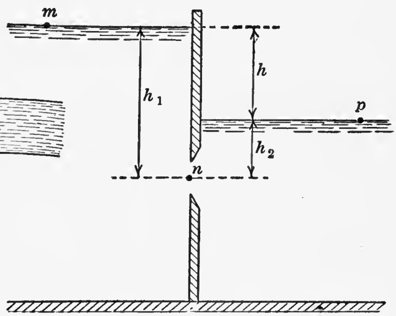

FIG. 68.—Submerged orifice.

coefficient of velocity therefore equals the coefficient of discharge. Thus $C_c = 1$ and $C_v = C$.

From experiments $C_v = 0.75$. The head lost, from equation (22) (Art. 54) is, therefore, $h_0 = 0.78\,\frac{v^2}{2g}$.

This case is important since the entrance to a pipe, which projects into a body of water, may be considered as a re-entrant tube and the head lost at entrance to the pipe is taken as the head lost in a re-entrant tube (Art. 102).

64. Submerged Orifice.—An orifice discharging wholly under water (Fig. 68) is called a submerged orifice. The assumption is usually made that every filament of water passing through the orifice is being acted upon by a head, $h_1 - h_2 = h$, the difference in elevation of water surfaces. Based upon this assumption

and using the same nomenclature as for orifices with free discharge

$$v_t = \sqrt{2gh} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$v = C_v\sqrt{2gh} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and

$$Q = C_cC_va\sqrt{2gh} = Ca\sqrt{2gh}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Coefficients of discharge for sharp-edged, submerged orifices are very nearly the same as for similar orifices discharging into the air.

The assumption that h_2 is the pressure head on the center of the orifice at its lower side is not strictly true unless all of the velocity head, due to the velocity of the water leaving the orifice, is lost in friction and turbulence as the velocity is reduced to zero. It has been shown experimentally that less than 90 per cent of this velocity head may be lost. Assuming a loss of 90 per cent, the pressure head at the center of the orifice is $h_2 - 0.12\frac{v^2}{2g}$. The effect of this condition on the discharge may be investigated by writing Bernoulli's equation between m and n and n and p (Fig. 68). This matter is not of great importance in connection with submerged orifices, since the discrepancy resulting from the use of formula (18) is relatively small and the coefficient of discharge which is determined from experiments eliminates this source of error.

The loss of head sustained at the outlet of a pipe discharging into a body of still water is discussed in Art. 102. The conditions of discharge in this case are practically identical with those of the submerged orifice discussed above.

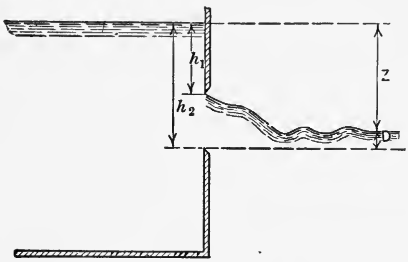

FIG. 69.—Partially submerged orifice.

65. Partially Submerged Orifices.—Fig. 69 represents a rectangular orifice, the bottom of which is submerged to a depth D. The upper and lower edges of the orifice are, respectively, h_1 and h_2 below the upper water surface. Z is the difference in elevation of water surfaces. L is the length of the

orifice. The total discharge through the orifice is evidently the combined discharge of the upper portion of the orifice discharging into the air and the lower portion discharging as a submerged orifice.

The theoretical formula for discharge from this orifice is interesting because of its relation to the submerged weir (Art. 79). Let Q_1 and Q_2 be, respectively, the discharges from the free and submerged portions of the orifice. Then from Art. 56, if C' is the coefficient of discharge for the upper portion

$$Q_1=\tfrac{2}{3}C'L\sqrt{2g}(Z^{3/2}-h_1^{3/2}), \qquad (42)$$

and by Art. 64, C'' being the coefficient of discharge for the lower portion

$$Q_2=C''L\sqrt{2gZ}(h_2-Z), \qquad (43)$$

and the total discharge, Q, for the orifice is

$$Q=Q_1+Q_2=L\sqrt{2g}[\tfrac{2}{3}C'(Z^{3/2}-h_1^{3/2})+C''\sqrt{Z}(h_2-Z)], \qquad (44)$$

or since $h_2-Z=D$,

$$Q=L\sqrt{2g}[\tfrac{2}{3}C'(Z^{3/2}-h_1^{3/2})+C''D\sqrt{Z}]. \qquad (45)$$

Since the coefficient of discharge for an orifice with free discharge is very nearly equal to the coefficient for a submerged orifice the equation may be put in the approximately equivalent form

$$Q=CL\sqrt{2g}[\tfrac{2}{3}(Z^{3/2}-h_1^{3/2})+D\sqrt{Z}]. \qquad (46)$$

If $h_1=0$ the orifice is a submerged weir and equation (45) becomes

$$Q=L\sqrt{2g}(\tfrac{2}{3}C'Z^{3/2}+C''D\sqrt{Z}). \qquad (47)$$

The submerged weir is discussed in Arts. 79 and 80.

66. Gates.—As used in engineering practice gates are forms of orifices. They may discharge freely into the air or be partially or wholly submerged. Though the principles underlying the discharge through orifices have been discussed in the preceding pages they cannot be applied accurately to gates because of the fact that gates do not ordinarily conform to the regular sections for which coefficients are directly available.

Fig. 70 illustrates a cross-section of a head gate such as is commonly used in diverting water from a river into a canal. A curtain wall extends between two piers, having grooves in which the gate slides. The bottom of the opening is flush with the floor of the structure. Such an opening has suppressed contraction at the bottom, nearly complete contraction at the top and partially suppressed contractions at the sides. Other equally complex conditions arise. The selection of coefficients for gates is therefore a matter requiring mature judgment and an intelligent use of the few available experimental data. Even the most

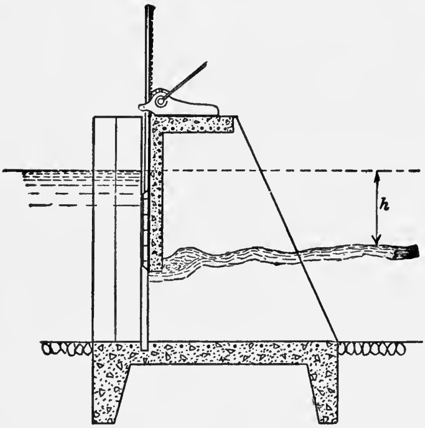

FIG. 70.—Headgate.

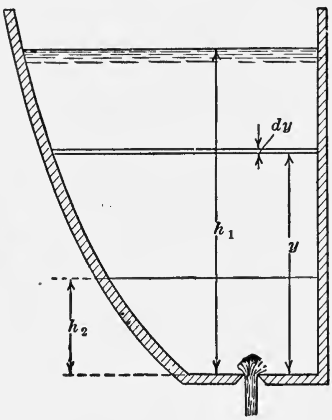

FIG. 71.—Discharge under falling head.

experienced engineers may expect errors of at least 10 per cent in the coefficients which they select and to provide for this uncertainty ample allowance should be made in designs.

67. Discharge under Falling Head.—A vessel is filled with water to a depth h_1 (Fig. 71). It is desired to determine the time required to lower the water surface to a depth h_2 through a given orifice. A is the area of the water surface when the depth of water is y and a is the area of the orifice. The rate of discharge at any instant when the head is y, the coefficient of discharge being C, is

$$Q=Ca\sqrt{2gy},$$

and in the infinitesimal time, dt, the corresponding volume of water which flows out is

$$dV = Ca\sqrt{2gy}\,dt.$$

In the same infinitesimal time the head will drop dy and the volume of water discharged will be

$$dV = A\,dy.$$

Equating the values of dV

$$A\,dy = Ca\sqrt{2gy}\,dt$$

or

$$dt = \frac{A\,dy}{Ca\sqrt{2gy}}. \qquad (48)$$

From this expression, by integrating with respect to y between the limits h_1 and h_2, the time required to lower the water surface the amount $(h_1 - h_2)$ may be determined or the time of emptying the vessel may be obtained by placing $h_2 = 0$, provided A can be expressed in terms of y. For a cylinder or prism the cross-sectional area, A, is constant and the formula after integration becomes

$$t = \frac{2A}{Ca\sqrt{2g}}(\sqrt{h_1} - \sqrt{h_2}). \qquad (49)$$

The above formulas apply also to vertical or inclined orifices, provided the water surface does not fall below the top of the orifice. The heads h_1 and h_2 are then measured to the center of the orifice. The time required to completely empty a vessel evidently can be determined only in the case of a horizontal orifice.

Example.—Two chambers, 1 and 2 (Fig. 72), with vertical sides, each chamber being 8 ft. wide, are separated by a partition. Chamber 1 is 25 ft. long and chamber 2 is 10 ft. long. At the bottom of the partition is an orifice 1 ft. by 2 ft. The orifice is at all times submerged. The coefficient of discharge is 0.85. At a certain instant the water surface is 10 ft. higher in chamber 1 than in chamber 2. After what interval

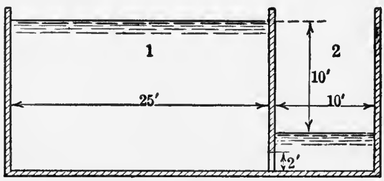

Fig. 72.

of time will the water surfaces in the two chambers be at the same elevation?

Solution.—Let y be the difference in elevation of water surfaces at any instant and dy be the change in the difference in elevation of water surfaces in time dt. The amount of water flowing into chamber 2 in time dt will be

$$dV = Ca\sqrt{2gy}\,dt = 0.85\times2\times8.02\sqrt{y}dt = 13.6\sqrt{y}dt.$$

Also in the same interval of time the head will drop $\frac{10}{35}\,dy$ in chamber 1 and rise $\frac{25}{35}\,dy$ in chamber 2. Then

$$dV = \frac{10\times25\times8}{35}\,dy = \frac{2000}{35}dy.$$

Equating values of dV

$$13.6\sqrt{y}dt = \frac{2000}{35}dy$$

or

$$dt = \frac{4.20dy}{\sqrt{y}}.$$

Integrating between the limits 10 and 0 and reducing,

$$t = 26.5 \text{ seconds.}$$

PROBLEMS

1. A sharp-edged orifice, 2 in. in diameter, in the vertical side of a large tank, discharges under a head of 16 ft. If C_c is 0.62 and C_v is 0.98 determine the diameter and velocity of the jet at the vena contracta and the discharge in cubic feet per second.

2. In Problem 1 how far from the vertical plane containing the orifice will the jet strike a horizontal plane which is 6 ft. below the center of the orifice?

3. A sharp-edged orifice, 3 in. in diameter, lies in a horizontal plane, the jet being directed upward. If the jet rises to a height of 21.2 ft. and the coefficient of velocity is 0.98, what is the depth of the orifice below the water surface, neglecting air friction. The pressure in the jet and on the surface of the reservoir is atmospheric.

4. In Problem 3, if $C_c = 0.62$, what is the diameter of the jet 16 ft. above the orifice?

5. If the orifice shown in Fig. 49, page 73, has a diameter of 2 in. and the diameter of the vena contracta is 1.6 in. determine the discharge if $h = 3.6$ ft., $V = 0$, $p_A = 9.7$ lbs. per square inch, $p_B = 1.3$ lbs. per square inch and the head lost is 0.8 ft.

6. A sharp-edged orifice, 4 in. in diameter, in the vertical wall of a tank discharges under a constant head of 4 ft. The volume of water discharged

in 2 minutes weighs 6352 lbs. At a point 2.57 ft. below the orifice the center of the jet is 6.28 ft. distant horizontally from the orifice. Determine C_c, C_v and C.

7. Determine the theoretical discharge (neglecting velocity of approach) from a vertical rectangular orifice 3 ft. long and 1 ft. high, the head on the top of the orifice being 2 ft.

8. A standard short tube, 4 in. in diameter, discharges under a head of 20 ft. What is the discharge in cubic feet per second? In gallons per day?

9. If a ½-in. hole is tapped into the standard short tube, referred to in Problem 8, at a point 2 in. from the entrance, determine the discharge through the tube, assuming the friction losses to remain the same.

10. If the upper end of a piezometer tube is connected with the ½-in. hole referred to in Problem 9 and the lower end is submerged in a pan of mercury, to what height will the mercury rise in the tube?

11. A Borda's mouthpiece 6 in. in diameter discharges under a head of 10 ft. What is the discharge in cubic feet per second? What is the diameter of the jet at the vena contracta?

12. Water is discharging through a gate 18 in. square. On the upstream side the water surface is 5 ft. above the top of the gate and on the downstream side it is 2 ft. above. If the coefficient of discharge is 0.82, what is the discharge in cubic feet per second?

13. A canal carrying 40 cu. ft. per second has a depth of water of 3 ft. A structure is built across the canal containing a gate 2 ft. square, the bottom of the gate being set flush with the bottom of the canal. If the coefficient of discharge is 0.85, what will be the depth of water on the upstream side of the gate?

14. If, in Problem 13, the gate has a width of 3 ft. and it is desired to increase the depth of water above the structure to 4 ft., what should be the height of the gate, all other conditions remaining the same?

15. A 3-in. fire hose discharges water through a nozzle having a diameter at the tip of 1 in. If there is no contraction of the jet and $C_v=0.97$, the gage pressure at the base of the nozzle being 60 lbs. per square inch, what is the discharge in gallons per minute?

16. In Problem 15 to what vertical height can the stream be thrown, neglecting air friction?

17. In Problem 15, if it is desired to throw a stream to a vertical height of 100 ft., what must be the pressure at the base of the nozzle?

18. In Problem 15 what is the maximum horizontal range (in the plane of the nozzle) to which the stream can be thrown?

19. A fire pump delivers water through a 6-in. main to a hydrant to which is connected a 3-in. hose, terminating in a 1-in. nozzle. The nozzle, for which $C_c=1$ and $C_v=0.97$, is 10 ft. above the hydrant and the hydrant is 50 ft. above the pump. What gage pressure at the pump is necessary to throw a stream 80 ft. vertically above the nozzle?

20. A cylindrical vessel 4 ft. in diameter and 6 ft. high has a sharp-edged circular orifice 2 in. in diameter in the bottom. If the vessel is filled with water how long will it take to lower the water surface 4 ft.?

21. A tank, which is the frustum of a cone having its bases horizontal

and axis vertical, is 10 ft. high and filled with water. It has a diameter of 8 ft. at the top and 3 ft. at the bottom. What is the time required to empty the tank through a sharp-edged orifice 3 in. square?

22. A hemispherical shell, with base horizontal and uppermost, is filled with water. If the radius is 8 ft. determine, the time required to empty through a sharp-edged orifice 6 in. in diameter located at the lowest point.

23. A tank 12 ft. long has its ends vertical, top and bottom horizontal, and is 6 ft. high. The top and bottom are rectangular, having widths of 8 ft. and 5 ft., respectively. A standard short tube, 4 in. in diameter, is located in one end near the bottom. If at the beginning the tank is full, find the time necessary to lower the water surface 4 ft.

24. In the tank described in Problem 23 assume that there is a vertical partition parallel with the ends and 5 ft. distant from one end. Near the bottom of this partition there is a circular, sharp-edged orifice 4 in. in diameter. If at the beginning the larger chamber is filled and the smaller chamber contains water having a depth of 2 ft., find the time required for the water surfaces to come to the same level.

CHAPTER VIII

FLOW OF WATER OVER WEIRS

68. Description and Definitions.—A weir may be described as any notch of regular form through which water flows. This notch may be in the side of a tank, reservoir or channel or it may be an overflow dam with retaining walls at its ends. In general any obstruction, having an approximately uniform cross-section, placed in a channel so that water must flow over it is a weir.

The edge or surface over which the water flows is called the *crest* of the weir. The overfalling sheet of water has been termed the *nappe.*

Weirs may be classified in two ways, (a) with reference to the shape of the notch and (b) with reference to the cross-sectional form of the crest.

Rectangular weirs—that is, weirs having a level crest and vertical sides, are the most generally used. Other weirs in more or less common use, named from the shape of the notch or opening, are *triangular weirs, trapezoidal weirs* and *parabolic weirs.*

Weir crests are constructed of many cross-sectional forms, but they all come under one of the general headings, (a) sharp-crested weirs, which are used primarily for the measurement of flowing water and (b) weirs not sharp crested which are used primarily as a part of hydraulic structures.

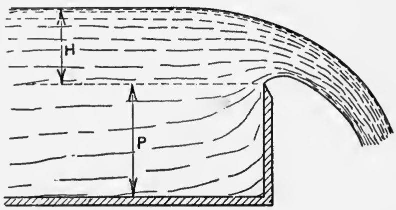

FIG. 73.—Sharp-crested weir.

A *sharp-crested weir* is a weir with a sharp upstream edge so formed that water in passing touches only this edge. The nappe from such a weir is contracted at its under side in the same way that the jet from a sharp-edged orifice is contracted. This is called *crest contraction.* If the sides of the notch also have sharp upstream edges so

that the nappe is contracted in width the weir is said to have *end contractions.* The nappe from a weir having a length equal to the width of the channel suffers no contraction in width and such a weir is said to have *end contractions suppressed.* Fig. 73 is a cross-section of a sharp-crested weir which illustrates crest contraction. Figs. 74 and 75 are views of weirs with end contractions. Fig. 76 shows a weir with end contractions suppressed.

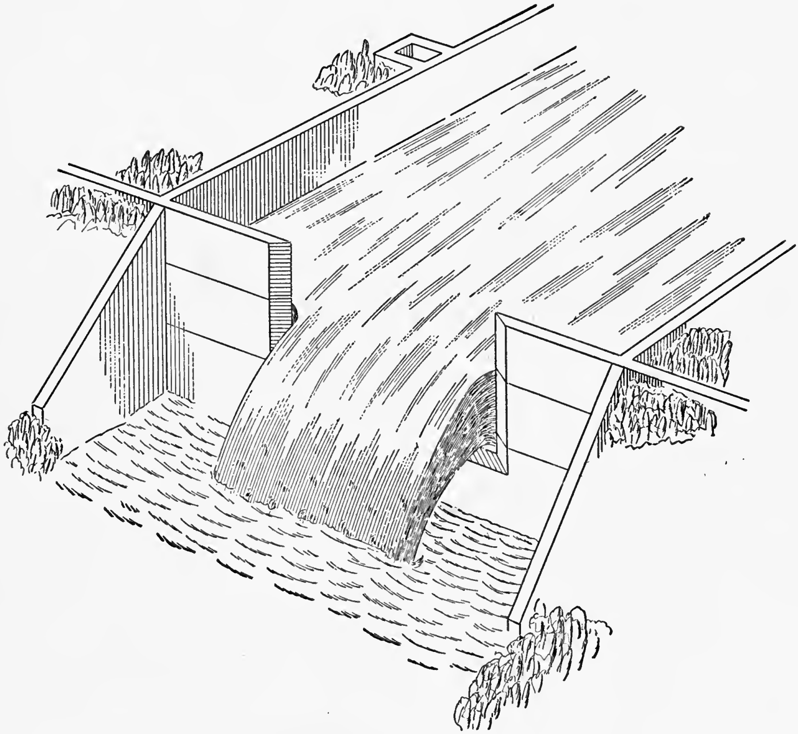

Fig. 74.—Weir with end contractions.

There is a downward curvature to the water surface near the weir crest (Fig. 73). This is called the *surface contraction.* The *head,* H (Fig. 73), is the vertical distance from the water surface, back of the effects of surface contraction, to the crest of the weir. The curvature of the water surface is not perceptible beyond a distance of about $2H$ upstream from the weir. The head is usually measured at distances of 6 to 16 ft. upstream from the weir.

The *vertical contraction* of the nappe includes both the surface contraction and the crest contraction. The section where the effects of crest contraction disappear, corresponding to the vena contracta of the jet, will be referred to as the *contracted section* of the nappe.

Incomplete contraction of the nappe occurs when the crest of a weir is so near the bottom, or the ends of a weir with end contractions are so near to the sides of the channel, as to interfere with the approach of the water filaments in directions parallel to the face of the weir. The conditions are similar to those causing partial suppression of the contraction of the jet issuing from an orifice, discussed in Art. 57.

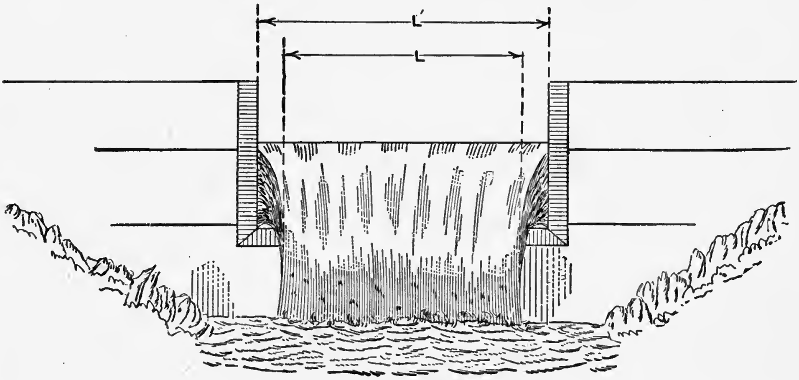

Fig. 75.—Weir with end contractions.

Weirs not sharp crested are constructed in a wide variety of cross-sectional forms as is exemplified in the many shapes of overflow dams now in existence. Such weirs have surface contraction similar to sharp-crested weirs, but conditions at the crest are different and vary with the sectional form (see Figs. 84 to 87). A variety of cross-sections of weirs of this class are shown in Fig. 88.

The term *velocity of approach*, as used in connection with weirs, means the mean velocity in the channel just upstream from the weir. The portion of the channel near where the head is measured is designated the *channel of approach*. The *height* of a weir, P (Fig. 73), is the vertical distance of the crest above the bottom of the channel of approach.

69. Velocity at any Depth.—Consider water to be discharging over the weir crest A (Fig. 77). In the derivation of the fundamental formula it will be assumed that the water flows without friction and also that there is no contraction of the nappe and therefore no pressure within the nappe. In order to write a general expression applicable to all filaments it will be necessary to make the further assumption that all of the water particles in

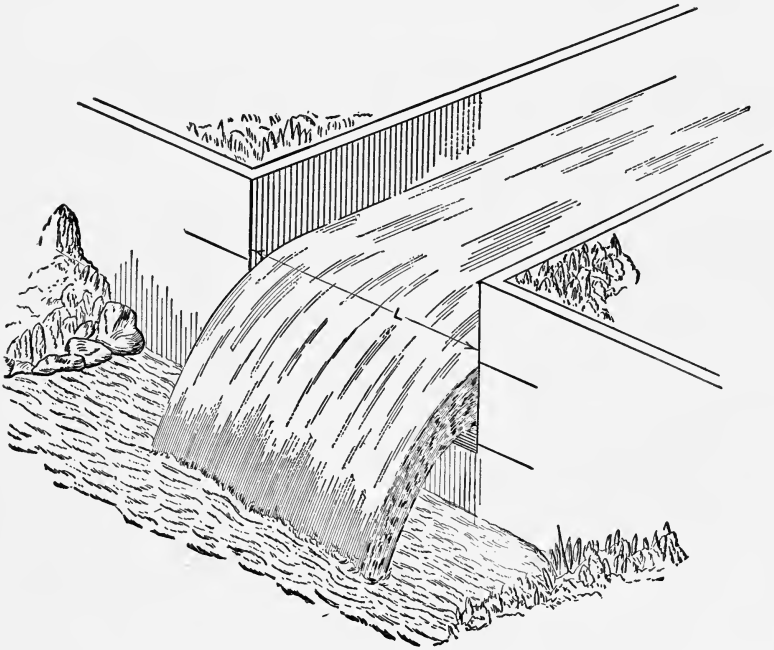

Fig. 76.—Weir with end contractions suppressed.

a cross-section of the channel of approach flow with the same velocity.

From the nature of the discharge over weirs it is evident that the water surface in the channel and the nappe must be subjected to the same pressure from surrounding gases, which is usually atmospheric pressure. All pressures excepting those resulting from the weight of the water may therefore be neglected.

The flowing water may be considered to be made up of filaments of which mn is one, m being a point in the channel of approach

and n a point in the nappe, in the plane of the weir. The filament passes over the weir at a distance y below the surface of the water. The point m is a distance h_m below the water surface and a distance z below n. v_m is the velocity at m and v_n is the velocity at n. Bernoulli's equation may be written between the points m and n as follows:

$$\frac{v_m^2}{2g}+h_m-z=\frac{v_n^2}{2g} \qquad (1)$$

and since $h_m-z=y$

$$y=\frac{v_n^2}{2g}-\frac{v_m^2}{2g} \qquad (2)$$

and

$$v_n=\sqrt{2g\left(y+\frac{v_m^2}{2g}\right)}. \qquad (3)$$

These formulas express the theoretical relation between depth and velocity for any point in the plane of the weir.

Introducing the assumption that all of the water particles in a cross-section of the channel of approach flow with the same velocity, V; v_m in formulas (2) and (3) may be replaced by V, which gives

$$y=\frac{v_n^2}{2g}-\frac{V^2}{2g}, \qquad (4)$$

and

$$v_n=\sqrt{2g\left(y+\frac{V^2}{2g}\right)}. \qquad (5)$$

If the cross-sectional area of the channel of approach is very much larger than the area of the notch, the velocity of approach is small and V may be called zero. The depth, y, at which the velocity v_n occurs is then from formula (4)

$$y=\frac{v_n^2}{2g}. \qquad (6)$$

and the theoretical velocity at a depth y, from formula (5) is

$$v_n=\sqrt{2gy}. \qquad (7)$$

70. Theoretical Formulas for Discharge.—Referring again to Fig. 77, let an origin be assumed at O, a distance H vertically

above the weir crest A. Formula (5) as derived in the preceding article is

$$v_n=\sqrt{2g\left(y+\frac{V^2}{2g}\right)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

This is also the equation of a parabola whose axis is the line OA and whose intersection with the axis is at M, a distance $\frac{V^2}{2g}$ above the origin. Assuming the curve MN to be the graph of the

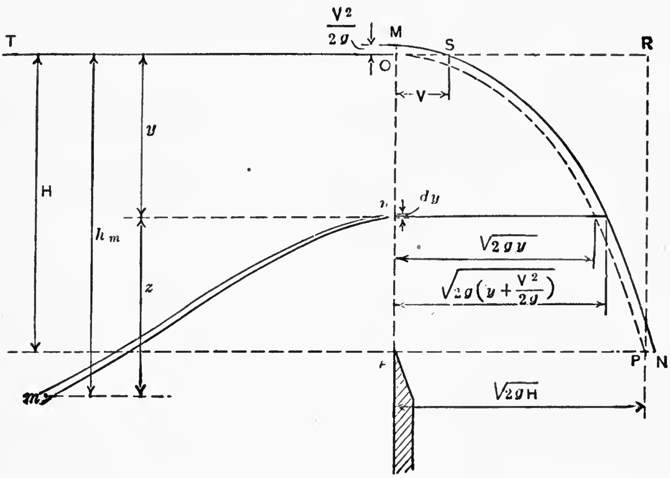

FIG. 77.

equation, the abscissa at any depth, y, is the theoretical velocity at this depth.

Considering a unit length of weir, the area of an elementary strip is dy and the theoretical discharge through this strip is

$$dQ_1=v_n dy. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

Substituting the value of v_n from equation (5)

$$dQ_1=\sqrt{2g\left(y+\frac{V^2}{2g}\right)}dy$$

and

$$Q_1=\sqrt{2g}\int^{H}\sqrt{y+\frac{V^2}{2g}}dy$$

or

$$Q_1=\tfrac{2}{3}\sqrt{2g}\left[\left(H+\frac{V^2}{2g}\right)^{3/2}-\left(\frac{V^2}{2g}\right)^{3/2}\right]. \quad . \quad . \quad . \quad (9)$$

This formula expresses the theoretical discharge over a weir 1 ft. long, assuming uniform velocity in a cross-section of the channel of approach, and neglecting the effects of friction and contraction of the nappe. It evidently is also the area of the surface *OSNA* (Fig. 77).

As a matter of convenience the symbol h may be substituted for $\frac{V^2}{2g}$. Making this substitution the theoretical discharge for a weir of length L becomes

$$Q_t=\tfrac{2}{3}\sqrt{2g}L[(H+h)^{3/2}-h^{3/2}], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

which formula may be transposed to the form

$$Q_t=\tfrac{2}{3}\sqrt{2g}LH^{3/2}\left[\left(1+\frac{h}{H}\right)^{3/2}-\left(\frac{h}{H}\right)^{3/2}\right]. \quad . \quad . \quad (11)$$

In this form the term within the brackets is the factor which corrects for velocity of approach. If the cross-sectional area of the channel of approach is large in comparison with the cross-sectional area of the nappe, the effect of velocity of approach will not be appreciable and may be considered to be zero. The above formulas then reduce to

$$Q_t=\tfrac{2}{3}\sqrt{2g}LH^{3/2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

which is the same as formula (27) (page 83).

This formula may also be derived directly from Fig. 77. The area of the surface *AOP* which represents the discharge over a weir 1 ft. long, being half of a parabolic segment, is equal to two-thirds of the area of the circumscribed rectangle *ORPA* or $\tfrac{2}{3}H\sqrt{2gH}$. The discharge for a weir of length L is therefore $\tfrac{2}{3}L\sqrt{2g}H^{3/2}$, which is the same as formula (12).

71. Theoretical Formula for Mean Velocity.—Since formula (9) which is the theoretical formula for discharge over a weir 1 ft. long is also an expression for the area of the surface *OSNA* and since the abscissas to this curve at any depth are the velocities at the depth, the mean of the abscissas between O and A gives the mean velocity of the water discharging over the weir. The

mean velocity is therefore the area of the surface $OSNA$ divided by H, and the expression for theoretical mean velocity is obtained by dividing formula (9) by H, which gives, after substituting h for $\frac{V^2}{2g}$.

$$v_t=\frac{Q_1}{H}=\tfrac{2}{3}\sqrt{2gH}\left[\left(1+\frac{h}{H}\right)^{3/2}-\left(\frac{h}{H}\right)^{3/2}\right]. \quad . \quad . \quad . \quad (13)$$

If the velocity of approach is considered to be zero, h also becomes zero and the above formula reduces to

$$v_t=\tfrac{2}{3}\sqrt{2gH}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Equating the right-hand members of equations (7) and (14) gives the theoretical depth at which the mean velocity occurs, or

$$y=\tfrac{4}{9}H. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

72. Weir Coefficients.—The assumptions which were made in the derivation of formula (10) may be summarized briefly as follows:

(*a*) All water particles in a cross-section of the channel of approach flow with the same velocity.

(*b*) There is no contraction of the nappe.

(*c*) The water flows without friction.

Since these conditions do not in reality exist, it is necessary to modify formula (10) and the formulas derived therefrom to make them applicable to actual conditions. To accomplish this, three empirical coefficients are applied to the formula, there being one coefficient to correct for the difference between assumed conditions and actual conditions for each of the above assumptions. The method of correcting for each assumption will be discussed in the order given above.

(*a*) *Correction for non-uniformity of velocity in cross-section of channel of approach.* The velocity in any cross-section of a channel is never uniform. As a result of the combined effects of friction, viscosity and surface tension (Arts. 7 and 110) velocities are lowest near the sides and bottom of an open channel and, if the channel is straight and uniform, the maximum velocity is below the surface and near the center of the channel. If there are no obstructions, velocities in a vertical line (Art. 110) vary approximately as the abscissas to a parabola. In the channel of approach where a weir obstructs the flow, the law of distribution

of velocities is not well understood and in cases where these velocities have been measured they have been found to vary quite irregularly. It is not practicable therefore to determine by analysis the extent to which discharges over a weir may be affected by the distribution of velocities in a cross-section of the channel of approach, but the general effect may be seen by studying certain assumed conditions.

Let the curve CMB (Fig. 78) represent any vertical distribution of velocities in the channel of approach for a strip of water

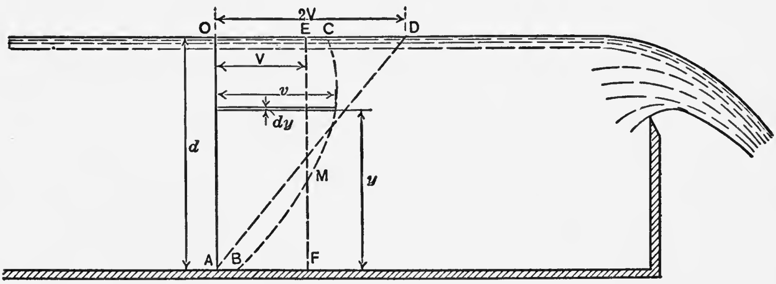

FIG. 78.—Velocities in channel of approach.

1 ft. wide. The velocity at any distance, y, above the bottom is v, the total depth being d. The kinetic energy for this strip of water is

$$KE=\int_0^d \frac{wv^3}{2g}dy=\frac{w}{2g}\int_0^d v^3dy. \quad . \quad . \quad . \quad . \quad (16)$$

The kinetic energy for any distribution of velocities can be determined from this formula where v can be expressed in terms of y.

Three conditions of assumed velocities are illustrated in Fig. 78. Uniform velocities are indicated by the vertical line EF. Velocities decreasing uniformly downward with a bottom velocity of zero are illustrated by the line DA. The line CMB illustrates a parabolic distribution of velocities. As these three lines are drawn, the mean velocity, V, is the same for each case.

For uniform velocities, v in equation (16) is constant and equal to V. Substituting this value and integrating, there results

$$KE=\frac{wdV^3}{2g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

For uniformly varying velocities, illustrated by the line DA (Fig. 78) $v=2V\frac{y}{d}$. Substituting this value in equation (16) and integrating

$$KE=\frac{wdV^3}{g}, \quad \ldots \ldots \ldots \quad (18)$$

which shows the kinetic energy for this distribution of velocities to be twice as great as for uniform velocity.

It may be shown also by writing the equation of a curve similar to CMB, v being expressed as a function of y, and substituted for v in equation (16), that the kinetic energy for this distribution of velocities is about 1.3 times the kinetic energy for uniformly distributed velocities.

Similarly for any variation in velocities in the cross-section of a channel, it may be shown that the water contains more kinetic energy if the velocity is non-uniform than if it is uniform.

In general, the kinetic energy contained in the water in the channel of approach may be written

$$KE=\alpha W\frac{V^2}{2g}, \quad \ldots \ldots \ldots \quad (19)$$

in which α is an empirical coefficient always greater than unity, and since velocity head is the kinetic energy contained in 1 lb. of water (Art. 43) the general expression for velocity head due to velocity of approach is $\alpha\frac{V^2}{2g}$ or αh.

This expression should, therefore, be written for h in formula (13), and calling v' the velocity after this correction has been applied

$$v'=\tfrac{2}{3}\sqrt{2gH}\left[\left(1+\frac{\alpha h}{H}\right)^{3/2}-\left(\frac{\alpha h}{H}\right)^{3/2}\right]. \quad \ldots \quad (20$$

(*b*) *Correction for contraction.*—The ratio of the thickness of the nappe at its contracted section to the head on the weir may be called the *coefficient of contraction*, C_c. This includes only vertical contraction, a separate correction being required for weirs with end contractions (see Art. 73). If t is the thickness of the nappe and H the head

$$C_c=\frac{t}{H} \quad \text{or} \quad t=C_cH,$$

and if v is the actual mean velocity in the contracted section of the nappe, L being the length of the weir, the discharge over the weir is

$$Q = tLv = C_c LHv. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

The average value of C_c for sharp-crested weirs is about 0.635.

(*c*) *Correction for friction.* The velocity in the nappe suffers a a retardation by reason of the combined effects of friction and viscosity. The ratio of the actual mean velocity, v, to the velocity v', which would exist without friction, is called the *coefficient of velocity.* Designating the coefficient of velocity by the symbol C_v,

$$C_v = \frac{v}{v'} \quad \text{or} \quad v = C_v v',$$

substituting this value of v in (21)

$$Q = C_c C_v LHv'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

The average value of C_v for a sharp-crested weir is probably about 0.98, the same as for a sharp-edged orifice.

Substituting the value of v' given in formula (20), the formula for discharge over a weir with the three corrective coefficients becomes

$$Q = \tfrac{2}{3}\sqrt{2g}\,C_c C_v L H^{3/2}\left[\left(1+\frac{\alpha h}{H}\right)^{3/2} - \left(\frac{\alpha h}{H}\right)^{3/2}\right]. \quad . \quad . \quad (23)$$

It is usual to combine $\frac{2}{3}\sqrt{2g}\,C_c C_v$ into a single coefficient, C, called the *weir coefficient,* then

$$C = \tfrac{2}{3}\sqrt{2g}\,C_c C_v. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

If $C_c = 0.635$ and $C_v = 0.98$, the values given above, $C = 3.33$, which is an average value of this coefficient. It is the value adopted by Francis as a result of his experiments on sharp-crested weirs.

Later experiments by Fteley and Stearns, and Bazin considered in connection with the Francis experiments, show quite conclusively that C is not a constant. Its value appears to be represented quite closely by the expression

$$C = \frac{3.34}{H^{0.03}}.$$

An investigation by Bazin gave the following value of C:

$$C=3.248+\frac{0.079}{H}.$$

With C substituted, formula (23) becomes

$$Q=CLH^{3/2}\left[\left(1+\frac{\alpha h}{H}\right)^{3/2}-\left(\frac{\alpha h}{H}\right)^{3/2}\right]. \quad . \quad . \quad . \quad . \qquad (25)$$

The expression within the brackets is the correction for velocity of approach. When the velocity of approach is so small that the head, h, due to this velocity may be considered zero, formula (25) becomes

$$Q=CLH^{3/2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (26)$$

Formula (25) includes coefficients which correct for all of the assumptions which were made in deriving the theoretical formula (10) for weirs with end contractions suppressed.

Formula (25) is often written in the equivalent form

$$Q=CL[(H+\alpha h)^{3/2}-(\alpha h)^{3/2}]. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (27)$$

73. Weirs with End Contractions.—The weir coefficient, C, does not include end contractions. A separate correction must therefore be applied to the above formulas to make them applicable to weirs with end contractions. End contractions reduce the effective length of a weir. Francis determined from his own experiments that the effective weir length is reduced an amount equal to $0.1H$ by each contraction. If L is the effective length of the weir, L' the measured length and N the number of contractions, from Francis' determination (see Fig. 75)

$$L=L'-0.1NH. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (28)$$

For two end contractions $N=2$. If contraction is suppressed at one end $N=1$.

Some of the later experiments do not substantiate the results of Francis, but no general formula better than the above has been suggested. On account of uncertainty regarding the best method of correcting for end contractions, where they can be properly used, weirs with end contractions suppressed are preferable.

74. Modifications of Fundamental Formula.—In the form given, formula (25) or its equivalent (27) is cumbersome and not

convenient to use as a base formula. It is therefore seldom used without modification. Francis adopted this formula without correcting for non-uniform velocity in a cross-section of the channel of approach, which gives α a value of unity, but this was before experiments on the effect of velocity of approach were available.

A common method of simplification is to simply drop the last term of formula (27) using as the base formula

$$Q=CL(H+\alpha h)^{3/2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

The value of the term $(\alpha h)^{3/2}$ which is dropped is represented by the area MOS (Fig. 77). The amount which the discharge is affected by the term αh as retained is represented by the area $OSNP$. These areas are purely illustrative as actual areas are dependent upon values of H and V. By substituting numerical values, however, it may be shown that within the range of conditions occurring in practice, the simplified formula is nearly equivalent to the original expression. It should also be noted that a large portion of the error that would otherwise be introduced by dropping the term $(\alpha h)^{3/2}$ may be corrected in the selection of coefficients. The present understanding of weir hydraulics and the experimental data available for the determination of empirical coefficients are not sufficient to justify too close an adherence to fundamental formulas.

Equation (29) is not in a form convenient to use since h depends upon V and therefore upon Q for its value. When Q is unknown a formula of this form must be solved by first determining the approximate value of Q, neglecting velocity of approach (formula 26). From this value of Q an approximate value of h may be obtained, which substituted in the formula involving velocity of approach correction gives a value of Q which is usually close enough for the purpose. If a closer result is desired the computations may be repeated using this new value of Q for determining h. This formula may be modified by mathematical transformation so that terms depending upon Q for their value do not occur on the right-hand side of the equation.

75. Algebraic Transformation of Formula.—The fundamental formula as derived in Art. 72 is

$$Q=CLH^{3/2}\left[\left(1+\frac{\alpha h}{H}\right)^{3/2}-\left(\frac{\alpha h}{H}\right)^{3/2}\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Expanding the left-hand term within the brackets by the binomial theorem gives a diminishing series, and dropping the terms of higher powers the equation becomes

$$Q=CLH^{3/2}\left[1+\frac{3}{2}\frac{\alpha h}{H}+\frac{3}{8}\left(\frac{\alpha h}{H}\right)^2 \ldots -\left(\frac{\alpha h}{H}\right)^{3/2}\right]. \quad . \quad . \quad (30)$$

An expression approximately equivalent to the above is obtained by dropping all of the terms within the brackets excepting the first two, which gives

$$Q=CLH^{3/2}\left(1+\frac{3\alpha}{2}\frac{h}{H}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

As explained in the preceding article, the value of equation (30) is changed but little by dropping the term $\left(\frac{\alpha h}{H}\right)^{3/2}$ and since the sum of the terms of the expanded series which are dropped is of opposite sign and less than $\left(\frac{\alpha h}{H}\right)^{3/2}$, formula (31), is a closer approximation to the fundamental formula (27) than formula (29).

It is now desired to eliminate h, which depends upon Q for its value. By definition of velocity of approach,

$$V=\frac{Q}{A}=\frac{CLH^{3/2}}{A} \text{ (approximately)}, \quad . \quad . \quad . \quad (32)$$

where A is the cross-sectional area of the stream in the channel of approach. The value of Q as substituted is an approximation since it does not include the velocity of approach correction. It is to be applied, however, to a term which is itself a small correction, making the error introduced by this approximation relatively unimportant. Using this value of V

$$h=\frac{V^2}{2g}=\frac{C^2L^2H^3}{2gA^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Substituting this value of h in formula (31) and reducing,

$$Q=CLH^{3/2}\left[1+\frac{3\alpha C^2}{4g}\left(\frac{LH}{A}\right)^2\right]. \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Replacing $\frac{3\alpha C^2}{4g}$ by a single coefficient, C_1, the formula becomes

$$Q=CLH^{3/2}\left[1+C_1\left(\frac{LH}{A}\right)^2\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

This is a convenient base formula for discharge over weirs. Some weir formulas are expressed in this form and most of the others may be readily reduced to it. This form of formula provides a direct solution for Q while other forms require a trial and error method of correcting for velocity of approach. The values of C and C_1 must be determined from experiments and at this point rational reasoning must give place to empirical science.

76. Weir Experiments.—Working formulas are obtained by determining experimentally the values of coefficients to be applied to derived formulas as, for instance, C and C_1 in (35). Many experiments, on sharp-crested weirs, covering a wide range of conditions of flow, have been performed during the past century. The most important of these are the experiments by Francis in 1852, those by Fteley and Stearns in 1877 and the Bazin experiments in 1886.[1] There are some inconsistencies in the results of the various experiments, but in general they substantiate the correctness of the reasoning in the preceding pages and the base formula derived thereby.

77. Formulas for Sharp-crested Weirs.—A large number of formulas for sharp-crested weirs have been published, but only those best known or those appearing to possess the greatest merit will be given.

The Francis Formula which is obtained by putting $C=3.33$ and $\alpha=1$ in formula (27) is as follows:

$$Q=3.33L[(H+h)^{3/2}-h^{3/2}] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

Substituting $C=3.33$ and $\alpha=1$ in (34) there is obtained the following formula which gives results very nearly the same as formula (36).

$$Q=3.33LH^{3/2}\left[1+0.26\left(\frac{LH}{A}\right)^2\right]. \quad . \quad . \quad . \quad (37)$$

This may be considered as another form of the Francis formula, more convenient than the original, since it affords a direct solution

[1] J. B. Francis: Lowell Hydraulic Experiments (4th edition, 1883). Also *Trans.* Amer. Soc. Civ. Eng., vol. 13, p. 303.. Fteley and Stearns: Flow of Water over Weirs. *Trans.* Amer. Soc. Civ. Eng., vol. 12 (1883). H. Bazin: *Annales des Ponts et Chaussées,* October, 1888. Translation by Marichal and Trautwine: *Proc.* Eng. Club, Phila., January, 1890. Also *Annales des Ponts et Chaussées* for 1894, 1er Trimestre.

for Q, whereas the form (36) requires the trial method of solution. The second term within the brackets is the velocity of approach correction. When the velocity of approach is very small the term $0.26\left(\frac{LH}{A}\right)^2$ may be neglected and the Francis formula reduces to

$$Q=3.33LH^{3/2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (38)$$

If there are end contractions the measured length of weir should be corrected by formula (28).

The Fteley and Stearns Formula,, based upon a study of their own experiments and the experiments of Francis, is

$$Q=3.31L(H+\alpha h)^{3/2}+0.007L, \quad . \quad . \quad . \quad . \qquad (39)$$

in which $\alpha=1.5$ for suppressed weirs and 2.05 for weirs with end contractions. Without velocity of approach correction $h=0$. Formula (28) is to be used to correct for end contractions

The Bazin Formula.—The experiments on suppressed weirs by Bazin covered a wide range of conditions. As a result of his investigation Bazin devised a formula applicable to suppressed weirs. As originally published, the Bazin formula is expressed in metric units. His base formula is of the form of (35). The value of coefficients which he derived (see Art. 72, page 111), expressed in English units, may be written

$$C=3.248+\frac{.079}{H}$$

and

$$C_1=0.55.$$

Substituting these values in (35) the formula becomes

$$Q=LH^{3/2}\left(3.248+\frac{.079}{H}\right)\left[1+0.55\left(\frac{LH}{A}\right)^2\right]. \quad . \quad . \quad . \qquad (40)$$

It was not intended by Bazin that this formula should apply to weirs with end contractions, though in the form given above it can be so used, after correcting the measured length, L', by formula (28). Without veloc'ty of approach correction the term within the brackets becomes unity.

For suppressed weirs in rectangular channels where L equals the width of the channel as well as the length of the weir and d equals the depth of water in the channel of approach, the area of

water section in the channel of approach equals Ld and formula (40) becomes

$$Q=LH^{3/2}\left(3.248+\frac{.079}{H}\right)\left(1+0.55\frac{H^2}{d^2}\right). \quad . \quad . \quad (41)$$

In this form the Bazin formula applies more conveniently to suppressed weirs.

The King Formula [1] is based upon a study of the experiments of Francis, Fteley and Stearns, and Bazin, from which (see Art. 72, page 111) values of

$$C=\frac{3.34}{H^{0.03}}$$

and

$$C_1=0.56$$

were obtained. Substituting these values the formula becomes

$$Q=3.34LH^{1.47}\left[1+0.56\left(\frac{LH}{A}\right)^2\right]. \quad . \quad . \quad . \quad (42)$$

This formula applies to weirs with and without end contractions. If there are end contractions the measured length of weir is to be corrected by formula (28). In rectangular channels where the weir length equals the width of channel, and d equals the depth of water in the channel of approach, formula (42) reduces to

$$Q=3.34LH^{1.47}\left(1+0.56\frac{H^2}{d^2}\right). \quad . \quad . \quad . \quad . \quad . \quad (43)$$

Without velocity of approach, the term within the parentheses equals unity and the formula becomes

$$Q=3.34LH^{1.47}. \quad . \quad (44)$$

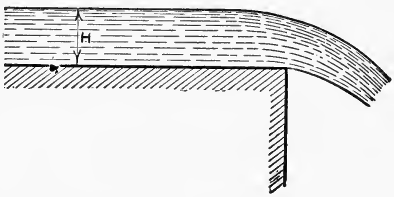

FIG. 79.—Fall.

Falls (Fig. 79) may be considered as weirs having a height of zero. In this case the head equals the depth of water,

[1] H. W. KING: Handbook of Hydraulics, McGraw-Hill Book Co., p. 71 (1918).

and if the channel has vertical sides, H/d, formulas (41) and (43), equals unity. These formulas then become, respectively,

$$Q=\left(5.04+\frac{0.122}{H}\right)LH^{3/2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

and

$$Q=5.21LH^{1.47}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

Both of these formulas lack experimental verification.

78. Discussion of Weir Formulas.—As is the case with all empirical formulas, weir formulas are no more accurate than the data upon which they are based. The three sets of experiments (see Art. 76) on which the above formulas are based give results which are somewhat conflicting so that no formula can agree with them all. The formula of Fteley and Stearns being based largely upon the results of their own experiments gives discharges somewhat less than the Bazin and King formulas which agree more closely with the Bazin experiments.

The Francis formula is based entirely upon the Francis experiments, which do not cover a wide range of conditions nor include any measurements for a determination of the effects of velocity of approach. As a result of this the Francis formula gives results considerably in error for high velocities of approach. It has also been found that the Francis formula gives too small discharges for low heads. For these reasons the Francis formula should not be considered as generally applicable to all conditions.

In general, formulas of the form of (35) which do not have terms dependent upon Q on the right-hand side of the equation are much more convenient to use than those of the form of (27) or (29). There is nothing sacrificed in accuracy by using formulas of the former type.

79. Submerged Weirs.—If the elevation of the water surface in the channel below a weir is higher than the crest of the weir the weir is said to be *submerged* or *drowned*. The water flowing away from the weir is sometimes called the *tail water*. The channel below the weir is called the *channel of retreat* and the velocity in this channel is the *velocity of retreat*. The *depth of submergence* is the difference in elevation between the tail-water surface and the crest of the weir. Other terms used correspond to those for weirs with free overfall.

Fig. 80 represents a submerged rectangular weir. The head is H and the depth of submergence D. The difference in elevation of water surfaces is $Z=H-D$. The length of weir is L.

In Art. 65 a submerged weir is shown to be a special case of a partially submerged orifice. The discharge may be considered as the combined discharge of a weir whose crest is at the elevation of the tail water and a submerged orifice, each discharging under a head Z. Neglecting velocity of approach, the combined discharge is

$$Q=\tfrac{2}{3}C'\sqrt{2g}LZ^{3/2}+C''\sqrt{2g}LD\sqrt{Z}. \quad . \quad . \quad . \quad (47)$$

Writing C, for $\tfrac{2}{3}C'\sqrt{2g}$ and C_2 for $C''\sqrt{2g}$ the formula becomes

$$Q=L(C_1Z^{3/2}+C_2D\sqrt{Z}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

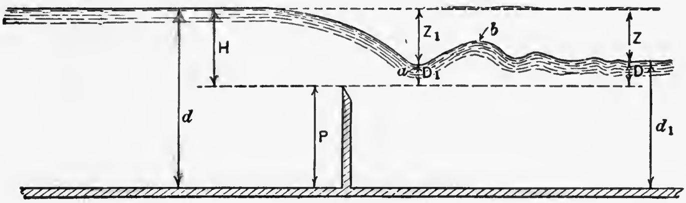

Fig. 80.—Submerged weir.

From experiments by Fteley and Stearns and by Francis are obtained values of coefficients which substituted in formula (48) give the following formula for submerged weirs:

$$Q=L\sqrt{Z}(3.4Z+4.4D) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

This formula does not provide any method of correcting for velocity of approach nor of making other corrections explained below. Results obtained by formulas of this type must be considered very approximate excepting for weirs that nearly duplicate the conditions of the original experiments.

Some investigators have considered C' and C'', formula (47), to be of the same value—that is, they have considered the crest contraction to equal the surface contraction. If C' and C'' are made equal, equation (47) may be reduced to the form

$$Q=CL\sqrt{Z}\left(H+\frac{D}{2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

This is the formula adopted by Fteley and Stearns. Accompanying the formula they give a table of values of C varying from 3.372 to 3.089 for different values of D/H. Since formula (50) requires an accompanying table of coefficients, it is not as convenient as formula (49) and possesses no advantages from considerations of accuracy.

80. Further Discussion of Submerged Weirs.—The discharge over submerged weirs is affected by velocity of approach in a manner very similar to the discharge over weirs with free overfall, but as the formula is more complicated, the application of a velocity of approach correction is more difficult. If in deriving formula (45) (page 95) the head is increased by an amount αh to correct for velocity of approach and the distance from the top of the orifice to the water surface is made zero the following formula is obtained:

$$Q=\tfrac{2}{3}C'L\sqrt{2g}[(Z+\alpha h)^{3/2}-(\alpha h)^{3/2}]+C''L\sqrt{2g}(Z+\alpha h)^{1/2}(H-Z). \quad (51)$$

This formula is complicated and is not reducible to a form permitting of simple application to submerged weir problems. With the limited experimental data available it is not possible to obtain values of the coefficients C', C'' and α with any degree of accuracy. There are required, moreover, other corrections, largely empirical in character which if applied to the above formula will still further complicate it.

The discharge over a submerged weir is greatly affected by conditions in the channel below the weir. Water flows over the crest of the weir at a velocity which is usually higher than the normal velocity of the tail water and a portion of this velocity is retained temporarily after leaving the weir. Where the slope of the channel is not sufficient to maintain this high velocity there is a piling-up effect just below the faster-moving water. This condition is illustrated in Fig. 80. The water has a higher velocity at a and a lower velocity at b than the normal velocity in the channel. This condition produces what is known as the *standing wave*, a being the trough and b the crest of the standing wave. Below the main wave a series of smaller waves form, which gradually reduce in size and finally disappear.

The factors affecting the height of the standing wave are not well determined, but from a purely empirical investigation it

appears to increase directly as the square root of H, D and Z (Fig. 80) and inversely as the square root of d_1. The depth of submergence, D, is one of the terms entering into submerged weir formulas and its value should be accurately determined. Theoretically, it is the depth, D_1, in the trough of the standing wave, that is to be used in formula (47), but usually it is the depth of submergence at D, below all turbulence caused by water flowing over the weir that is most easily measured and most convenient to use in submerged weir problems.

The only experiments on submerged weirs that furnish any data relative to the effect of velocity of approach and channel conditions below the weir are the experiments by Bazin. Two formulas for submerged rectangular sharp-crested weirs without end contractions which accord very well with the results of the Bazin experiments are given below. These formulas are largely empirical in character and a discussion of their derivation is not given. The symbols used are indicated in Fig. 80. The submerged weir formula by Bazin[1] may be written

h

$$Q=LH^{3/2}\left(3.248+\frac{0.79}{H}\right)\left(1+0.55\frac{H^2}{d^2}\right)\left(1.05+0.21\frac{D}{P}\right)\sqrt[3]{\frac{Z}{H}}. \quad (52)$$

h'

The submerged weir formula by King[2] is

$$Q=3.34LZ^{1.47}\left(1+0.56\frac{H^2}{d^2}\right)\left(1+\frac{1}{5}\sqrt{\frac{HD}{d_1Z}}\right)\left(1+1.2\frac{D}{Z}\right). \quad (53)$$

Each of the above formulas will require further experimental verification before it can be considered applicable to all conditions.

81. Triangular Weirs.—Fig. 81 represents a triangular notch or weir over which water is flowing. The measured head is H and the distance between the sides of the weir at the water surface is l. The sides make equal angles with the vertical.

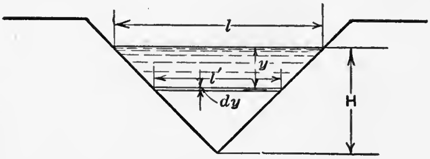

FIG. 81.—Triangular weir.

The area of an elementary horizontal strip dy in thickness is

[1] *Annales des Ponts et Chaussées* for 1898, 1er Trimestre, p. 235.

[2] Handbook of Hydraulics, p. 82.

$l'dy$. Neglecting velocity of approach, the theoretical velocity through this strip for a head y is $\sqrt{2gy}$ and the theoretical discharge is

$$dQ_t = l'\sqrt{2gy}dy$$

from similar triangles

$$l' = \frac{H-y}{H}l.$$

Combining the two equations

$$dQ_t = l\sqrt{2g}\frac{(H-y)\sqrt{y}}{H}dy,$$

integrating between the limits H and 0 and reducing,

$$Q_t = \tfrac{4}{15}\sqrt{2g}lH^{3/2}. \qquad (54)$$

The slope which the sides of the weir make with the vertical may be represented by z, then

$$\frac{\frac{1}{2}l}{H} = z \quad \text{or} \quad l = 2zH. \qquad (55)$$

Substituting this value of l in (54) the theoretical formula for discharge, expressed in terms of head and slope of sides, is,

$$Q_t = \tfrac{8}{15}\sqrt{2g}zH^{5/2}. \qquad (56)$$

Applying a coefficient of discharge and combining it with the constant terms the same as for rectangular weirs

$$Q = CzH^{5/2}, \qquad (57)$$

in which the value of C must be determined experimentally.

If the angle between the sides is a right angle, z equals unity. Most of the available experimental data are for right-angled notches. Triangular weirs having other angles are seldom used. Experiments indicate quite clearly that C is not a constant, its value decreasing with increasing heads.

The following are values of C as obtained from various sets of experiments[1] together with corresponding formulas for sharp-

[1] Prof. James Thompson: Experiments on Triangular Weirs. British Association Reports, 1861. James Barr: Flow of Water over Triangular Notches. Engineering, April 8 and 15, 1910. H. W. King: Handbook of Hydraulics, p. 86, 1918.

edged, right-angled, triangular weirs. From Thompson's experiments

$$C \text{ (mean)} = 2.54 \qquad Q = 2.54H^{5/2}. \quad . \quad . \quad . \quad . \qquad (58)$$

From Barr's experiments;

$$C = \frac{2.48}{H^{0.02}} \qquad Q = 2.48H^{2.48}. \quad . \quad . \quad . \qquad (59)$$

From experiments at the University of Michigan;

$$C = \frac{2.52}{H^{0.03}} \qquad Q = 2.52H^{2.47}. \quad . \quad . \quad . \qquad (60)$$

Other experiments on triangular weirs give results varying somewhat from those listed above.

One interesting fact brought out by Barr's experiments is that the discharge over a sharp-edged, metallic, triangular weir may be 2 per cent greater when the inner face of the metal is rough than when it is smooth. The rougher face, by retarding the movement of water parallel to it, reduces the velocities which have the greatest influence on contraction, thus reducing the contraction and so increasing the discharge.

The effect of velocity of approch on triangular weirs is similar in character to the effect on rectangular weirs. No data for determining coefficients are, however, available. From the nature of the triangular weir the cross-sectional area of the nappe is usually very much smaller than that of the channel of approach. The velocity of approach is therefore small and the error introduced by neglecting it is usually inappreciable. This has been confirmed experimentally by Barr.

The triangular weir affords an excellent method of measuring small discharges. Formula (59) probably applies more accurately to sharp-edged notches cut in very smooth metal and (60) to sharp-edged notches cut in rougher metal, such as ordinary commercial steel plate.

For angles slightly greater or less than 90° it is probable that the values of C listed above, if substituted in formula (57), will give quite accurate results.

82. Trapezoidal Weirs.—Fig. 82 represents a trapezoidal weir having a horizontal crest of length L. The sides are equally inclined, making angles $a/H = z$ with the vertical.

By writing the equation

$$dQ_t = l'\sqrt{2gy}dy$$

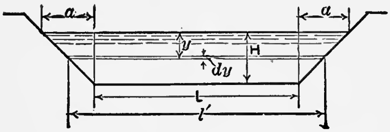

FIG. 82.—Trapezoidal weir.

and expressing l' in terms of y and known quantities in a manner similar to that used for triangular weirs, and integrating and reducing, the following formula for the theoretical discharge over trapezoidal weirs without velocity of approach correction can be obtained:

$$Q_t = \tfrac{2}{3}\sqrt{2g}LH^{3/2} + \tfrac{8}{15}\sqrt{2g}zH^{5/2}. \quad . \quad . \quad . \quad . \quad (61)$$

The same formula is obtained directly by the addition of the theoretical discharges over rectangular and triangular weirs. With coefficients included the formula for discharge may be written

$$Q = C_1LH^{3/2} + C_2zH^{5/2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

There are no experimental data for the determination of C_1 and C_2 and for this reason the trapezoidal weir has little practical value.

83. The Cippoletti Weir.—A trapezoidal weir, having a value of $z = a/H$ (Fig. 82) of $\frac{1}{4}$, is called a Cippoletti [1] weir. This slope of the sides is approximately that required to secure a discharge through the triangular portion of the weir opening that equals the decrease in discharge resulting from end contractions. The advantage of the Cippoletti weir is that it does not require a correction for end contractions. The method employed by Cippoletti in arriving at his value of z is as follows:

The discharge through the triangular portion of the weir, C' being the coefficient of discharge, is

$$Q = \tfrac{8}{15}C'\sqrt{2g}zH^{5/2}.$$

The decrease in discharge resulting from end contractions, C'' being the coefficient of discharge, according to Francis is

$$Q = \tfrac{2}{3}C''\sqrt{2g}\,0.2H^{5/2}.$$

Equating the right-hand members of these equations, assuming C' to equal C'', and reducing, there results

$$z = \tfrac{1}{4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

[1] C. CIPPOLETTI: Canal Villoresi (1887). Description of trapezoidal weir.

The formula given by Cippoletti for determining the discharge over Cippoletti weirs is

$$Q=3.367LH^{3/2}, \qquad (64)$$

to be corrected for velocity of approach by the Francis method. Later experiments indicate that this formula gives too great discharges for the higher heads when the velocity of approach is low. The Cippoletti weir is used quite extensively in western United States for measuring irrigation water where precision in measurement is not essential.

84. Weirs Not Sharp Crested.—Weirs having cross-sections such that they partially or completely suppress contractions at the

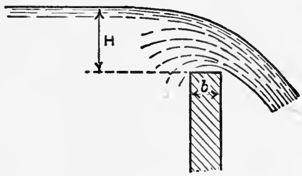

FIG. 83.—Weir with rectangular cross-section, Nappe springing clear.

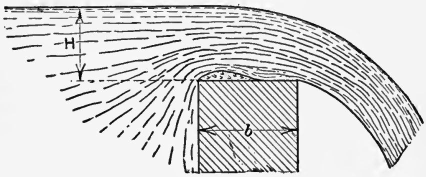

FIG. 84.—Weir with rectangular cross-section, Nappe adhering.

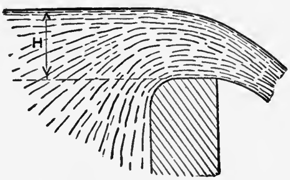

FIG. 85.—Weir with rounded crest.

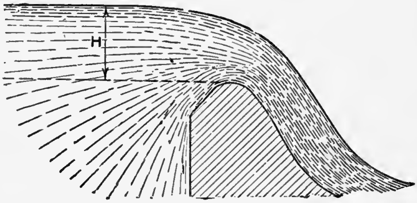

FIG. 86.—Weir with Ogee cross-section.

crest are used frequently in hydraulic structures. Spillway sections of dams are examples of this type of weirs. Such weirs also may be used as a means of measuring water if coefficients for the particular shape of weir are available.

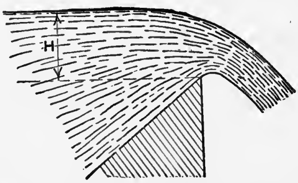

FIG. 87.—Weir with triangular cross-section.

Figs. 83 to 88 illustrate various cross-sections of weirs. Figs. 83 and 84 have rectangular sections with sharp upstream corners. If the breadth of weir, b (Fig. 83), is about $\frac{1}{2}H$ or less the nappe

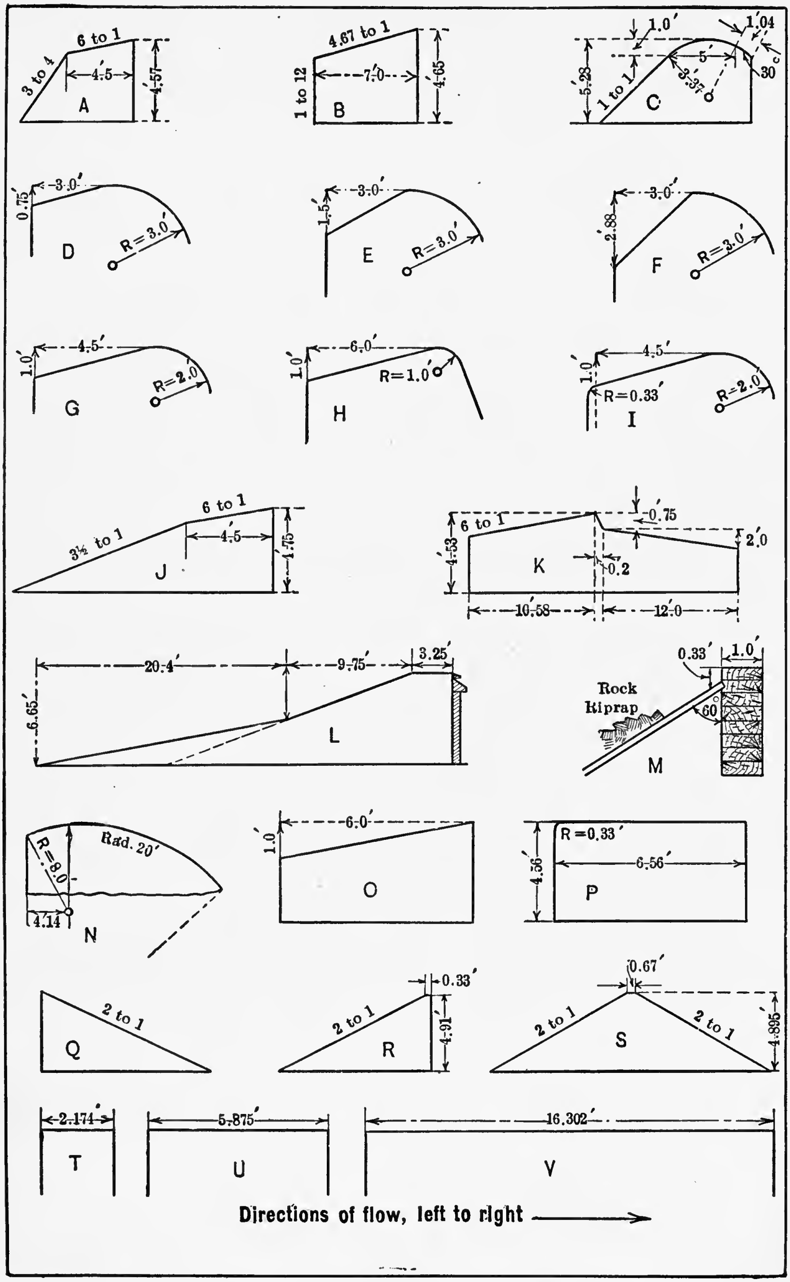

FIG. 88.—Various sections of weirs and dams.

Horton's Values of Weir Coefficient, C. (See Fig. 88)

Cross-section	Head in Feet, H									
	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
A	.....	3.46	3.45	3.42	3.35	3.32	3.33	3.37	3.41	3.44
B	.....	.....	.....	.....	3.43	3.39	3.38	3.38	3.39	3.40
C	.....	3.26	3.28	3.32	3.38	3.47	3.53	3.59	3.63	3.66
D	3.29	3.29	3.32	3.36	3.40	3.43	3.48	3.53	3.62	3.72
E	3.27	3.38	3.46	3.51	3.55	3.58	3.61	3.67	3.74	3.83
F	3.15	3.45	3.63	3.75	3.82	3.87	3.88	3.88		
G	3.18	3.30	3.38	3.42	3.46	3.49	3.52	3.53		
H	3.28	3.50	3.54	3.52	3.36	3.31	3.30	3.30		
I	3.18	3.27	3.43	3.52	3.59	3.64	3.68	3.70		
J	.....	.....	3.44	3.35	3.30	3.32	3.37	3.38	3.39	3.39
K	.....	3.12	3.20	3.22	3.22	3.22	3.22	3.22	3.22	3.22
L	3.12	3.14	3.10	3.14	3.20	3.26	3.21	3.36		
M	.....	3.80								
N	3.10	3.10	3.33							
O	3.53	3.54	3.54	3.49	3.35	3.27	3.25	3.25		
P	.....	2.81	2.81	2.81	2.81	2.81	2.81	2.81	2.81	2.81
Q	3.49	3.50	3.52							
R	.....	3.72	3.82	3.85	3.82	3.76	3.68	3.68	3.73	3.82
S	.....	.....	3.58	3.56	3.57	3.58	3.60	3.62	3.65	3.68
T	2.70	2.64	2.64	2.70	2.80	2.89				
U	2.72	2.64	2.64	2.64	2.64	2.64				
V	2.72	2.63	2.63	2.63	2.63	2.63	2.63	2.63	2.63	2.63

will spring clear of the downstream edge and there will be complete crest contraction. In this case the discharge will be given by the formula for sharp-crested weirs. If the breadth, b, is such that the nappe does not spring clear, as is indicated in Fig. 84, the free fall of the nappe is interfered with and the discharge is less than that of a sharp-crested weir. In Fig. 85 the upstream edge of the weir is rounded, which reduces crest contraction and thereby increases the discharge. By proper design the crest contraction may be reduced very nearly to zero.

The base formula commonly used for weirs not sharp crested (see Art. 72) is

$$Q=CLH^{3/2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

in which C is a coefficient varying with H whose value must be determined at different heads for each shape of crest.

After a thorough investigation Horton[1] has prepared tables and curves of C, corresponding to different heads, for practically all shapes of weir sections for which experimental data are available. In computing the values of his coefficients Horton assumed the velocity of approach correction given by the formula

$$Q=CL\left(H+\frac{V^2}{2g}\right)^{3/2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

This formula is obtained from formula (27) by giving α a value of unity and dropping the last term. The correction is doubtless too small, but since it was used in reducing the experimental data it should be applied in weir problems where Horton's coefficients are used. By substituting $\alpha=1$ in equation (34) and reducing there is obtained

$$Q=CLH^{3/2}\left[1+0.024C^2\left(\frac{LH}{A}\right)^2\right], \quad . \quad . \quad . \quad . \quad (66)$$

which gives results practically equivalent to (65) and is more convenient to use.

Fig. 88 shows various sections of weirs and dam crests for which experimental data are available. The table on page 127 gives Horton's values of C for these shapes.

The degree of accuracy which may be expected from the use of weirs not sharp crested depends upon the experimental data available for determining C. Inasmuch as there are innumerable shapes that may be used, it is not probable that experimental data for any large number of them will be secured for many years. Complete data for any particular shape of weir requires an exhaustive research similar to that required for sharp-crested weirs. The data at present available are, however, sufficient to assist in the selection of approximate coefficients for the shapes of weirs commonly used in hydraulic design.

Weirs not sharp crested, having cross-sections similar to the shapes for which experimental values of coefficients are available, may be used for the approximate measurement of discharges. There are some cross-sectional forms which might be more satisfactory for the measurement of flowing water than sharp-crested weirs if as complete experimental data for them were available.

[1] Robert E. Horton: Weir Experiments, Coefficients and Formulas. Water Supply and Irrigation Paper, No. 200, U. S. Geol. Survey (1907).

Existing dams frequently may be used for estimating flood discharges of streams where direct measurements of discharge by other methods are impracticable.

85. Broad-crested Weirs or Chutes.—A weir having a broad flat top such as is illustrated in Fig. 89 is called a *broad-crested weir.* A broad-crested weir is usually understood to be a weir having an approximately rectangular cross-section with a width of top, b (Figs. 84 and 89) great enough to prevent the nappe from springing clear of the top of the weir. Fig. 89 may also be considered to represent a longitudinal section of a chute, that is, a short channel discharging from a body of comparatively still water. The chute bears a relation to the weir analogous to the relation of the short tube to the orifice (page 85). A

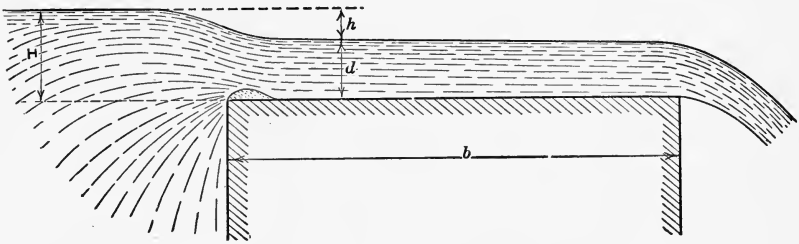

Fig. 89.—Broad-crested weir.

rational derivation of a formula for discharge over broad-crested weirs or chutes is given below.

There will be a drop, h, Fig. 89, in the water surface near the upstream edge of the crest. The velocity of water below this drop is that due to the head, h, or

$$v = C_v\sqrt{2gh},$$

where v is the mean velocity and C_v is the coefficient of velocity. If the top of the weir is level or has a very gentle slope the depth, d, will remain very nearly constant from the place where h is measured to the lower edge of the crest. With a greater slope of crest the velocity will accelerate and d will gradually decrease toward the lower edge of the crest. The discharge will not, however, be materially affected by the slope of the crest, provided it is sufficient to maintain the velocity v, since, as is shown below there is a maximum discharge which can not be exceeded.

If L is the length of weir or width of chute, the area through which water is discharging under a head, h, is

$$a=Ld=L(H-h).$$

The mean velocity is that due to the head, h, multiplied by the coefficient of velocity C_v, and the discharge is

$$Q=C_vL(H-h)\sqrt{2gh}. \qquad (67)$$

The coefficient of contraction is unity and C_v, therefore, is also the coefficient of discharge.

In this equation $Q=0$ when $h=0$ and also when $h=H$. There is therefore an intermediate value of h which makes Q a maximum. This maximum value of Q can be obtained by differentiating and equating to zero, which gives

$$\frac{dQ}{dh}=C_v\sqrt{2g}L\left(\frac{H-h}{2\sqrt{h}}-\sqrt{h}\right)=0,$$

whence,

$$h=\tfrac{1}{3}H. \qquad (68)$$

Substituting this value of h in equation (67) and reducing gives

$$Q=3.087C_vLH^{3/2}. \qquad (69)$$

The coefficient, C_v, is similar in character to the coefficient of velocity for a standard short tube, Fig. 60. Its value depends upon the shape of the upstream edge of the crest and probably approaches a maximum value of about 0.98 when this edge is so rounded as to prevent contraction. Formula (69) may also be written

$$Q=CLH^{3/2}, \qquad (26)$$

which is the base formula for weirs not sharp crested.

From experiments on broad-crested weirs it has been found that for weirs having a breadth of 10 ft. or more, discharging under a head of 1.0 ft. or more,

$$Q=2.63LH^{3/2}, \qquad (70)$$

which corresponds to a coefficient, C_v, in formula (69) of 0.85. If there are end contractions a separate correction must be applied to the length.

Formula (69) is of fundamental importance in connection with the entrance conditions for open channels. It gives the maximum rate at which water can be drawn through an open channel from any body of comparatively still water. The rate of discharge may be less but it can never be more than that given by the formula (see Art. 126).

86. Measurement of Head.—In using weirs to measure the rate of discharge, the head, length of weir and cross-sectional area of the channel of approach must be carefully measured. The last two of these usually need to be measured but once and can then be used in all subsequent determinations of Q.

The head is measured with some form of a gage which is set in a fixed position. The elevation of the zero of the gage with reference to the crest of the weir should be accurately determined. It is preferable to measure the head in a well or stilling box connected to the channel by a small pipe, the end of which is flush with the side of the channel. This provides a means for measuring the head in still water and reduces the effect of waves which are usually present in the channel of approach. For the most precise work a hook gage should be used.

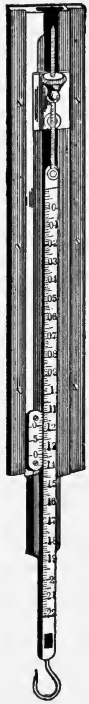

FIG. 90. Hook gage.

The *hook gage*, Fig. 90, consists of a graduated metallic rod with a pointed hook at the bottom which slides vertically in fixed supports. By means of a vernier attached to one of the supports, readings to thousandths of a foot may be taken. The rod usually has a range of movement of about 2 ft. The gage should be rigidly attached to a support at such an elevation that the movement of the hook covers the range of water surface elevations to be read. To take a reading, the point of the hook is lowered below the surface and then slowly raised by the screw at the top of the instrument. Just before the point of the hook pierces the skin of the water, a pimple is seen on the surface; the hook is then lowered slightly until the pimple is barely visible and the vernier is read.

Where less precision is required, especially for securing continuous records of elevation as in ordinary stream gaging work,

some other form of gage is desirable. An ordinary *staff gage,*—that is, a painted rod graduated to feet and decimals of a foot so set that the water surface comes in contact with the graduations—is quite satisfactory in some cases.

There are a great many different types of recording gages which give continuous records of water-surface elevation. These gages either provide a record by a graph, the coordinates of which indicate the time and stage, or by a device that prints elevations at stated intervals of time. The essential parts of a recording gage are: a float which rises and falls with the surface of the water, a device for transferring the vertical motion of the float to the record, a recording device, and a clock.

Another device for determining head is a plummet attached to the end of a steel tape. This is used to measure the vertical distance from a fixed point above the channel of approach to the water surface. The reading of the tape when the point of the plummet is at the elevation of the crest of the weir is first determined accurately and the difference between this reading and the reading when the point just touches the water surface gives the head on the weir. This method gives accurate results, but for precise work it probably is preferable to measure the head in a stilling box with a hook gage, so as to conform to the conditions of the experiments upon which weir formulas are based.

The head always should be measured far enough upstream from the weir to be well above the effects of surface contraction. In their experiments, Francis, and Fteley and Stearns measured heads 6 ft. and Bazin 16.4 ft. upstream from the weir. The distance selected should preferably conform approximately to that used in the experiments on which the formula to be used in computing discharges is based, though accurate comparative measurements show an almost imperceptible difference between heads measured 6 ft. and those measured 16.4 ft. from the weir.

87. Conditions for Accurate Measurement over Sharp-crested Weirs.—To obtain maximum accuracy the face of the weir should be vertical and the crest level. The crest should be cut from plate metal, true to line with a flat top and sharp upstream corner.

Suppressed weirs having a length equal to the channel width have a space below the nappe which may have no connection with the outside air. In passing over this space the nappe

carries with it all or a portion of the entrapped air, thus reducing the pressure underneath and causing the nappe to be depressed. This is equivalent to reducing crest contraction making the usual formulas inapplicable. The space under the nappe, therefore, should be connected by pipes or by other means with the outside air.

In general all conditions such as dimensions of weir and channel and ranges of head should conform as nearly as practicable to the conditions of the experiments which form the basis of the formula that is to be used in computing discharges. The length of the weir should be at least three times the measured head. Heads less than 0.2 ft. are undesirable since very low heads create a tendency for the nappe to adhere to the weir crest thus affecting the coefficient of contraction. Though it has not been definitely proved, it appears from rather limited experimental data that weir formulas apply as accurately for heads up to 4 ft. as for lower heads.

Weirs with end contractions should have their ends at a distance of at least two times the head from the sides of the channel in order to insure complete contraction.

PROBLEMS

1. A sharp-crested weir 4 ft. high extends across a rectangular channel 12 ft. wide. If the measured head is 1.22 ft., determine the discharge, using formulas (36), (37), (40) and (42).

2. Solve Problem 1, changing the height of weir to 2 ft. and the measured head to 1.54 ft.

3. Solve Problem 1, changing the height of weir to 2 ft. and the measured head to 0.25 ft.

4. What length of weir should be constructed in a stream 100 ft. wide so that the measured head will be 1.50 ft. when the discharge is 120 cu. ft. per second?

5. A rectangular channel 20 ft. wide has a 3-ft. depth of water flowing with a mean velocity of 2.45 ft. per second. Determine the height of sharp-crested suppressed weir that will increase the depth in the channel of approach to 5 ft.

6. A sharp-crested weir 2.5 ft. high is built across a rectangular flume 30 ft. wide. The measured head is 1.25 ft. In the flume is another sharp-crested weir having a height of 3.5 ft., the middle of the weir being on the center line of the flume. If the measured head on the latter weir is 1.62 ft. what is the length of crest?

7. A rectangular, sharp-crested weir is to be consructed in a stream in which the discharge varies from 2 cu. ft. per second to 50 cu. ft. per second.

Determine a length of weir, such that the measured head will never be less than 0.2 ft. nor greater than one-third of the length of weir.

8. Determine the discharge over a right-angled, triangular weir if the measured head is 1.82 ft.

9. The discharge over a right-angled, triangular weir is 7.28 cu. ft. per second. What is the measured head?

10. A channel is carrying 10 cu. ft. per second of water. Assuming that an error of 0.005 ft. may be made in measuring the head, determine the percentage of error resulting from the use of a right-angled, triangular weir, and also from the use of a rectangular weir 10 ft. long.

11. The measured discharge over a dam 100 ft. long is 520 cu. ft. per second when the head is 1.28 ft. Determine the weir coefficient for this head.

12. If in a certain channel the velocity varies uniformly from 3 ft. per second at the surface to 1 ft. per second at the bottom, determine the corresponding value of α.

13. An overflow masonry dam is to be constructed across a stream. The stream is estimated to have a maximum flood discharge of 30,000 cu. ft. per second, when the elevation of water surface at the dam site is 1132.0. Six sluice gates each 8 ft. high and 6 ft. wide ($C=0.85$) are to be constructed in the dam with their sills at elevation 1122.5. The main overflow weir for which $C=2.63$ will be 200 ft. long with a crest elevation of 1184.0. An auxiliary weir 600 ft. long with a crest elevation of 1185.3 will operate during floods. For this weir $C=3.40$. With all sluice gates open what will be the elevation of the water surface upstream from the weir when the discharge is 30,000 cu. ft. per second? Neglect velocity of approach.

14. A submerged sharp-crested weir 2.5 ft. high extends clear across a channel having vertical sides and a width of 10 ft. The depth of water in the channel of approach is 4.0 ft., and 35 ft. downstream from the weir the depth of water is 3.0 ft. Determine Q by formulas (52) and (53).

15. A channel 20 ft. wide with vertical sides is carrying 400 cu. ft. per second of water at a depth of 4.0 ft. How high a sharp-crested weir should be constructed across the channel to raise the elevation of the water surface 0.5 ft.?

CHAPTER IX

FLOW OF WATER THROUGH PIPES

88. Description and Definitions.—As the term is used in hydraulics, a *pipe* may be defined as a conduit which carries water under pressure. More commonly pipes are of circular cross-section, and hydraulic formulas for the flow of water through pipes are usually expressed in a form particularly adaptable to circular pipes, but the same general laws apply regardless of the cross-sectional shape of the pipe.

Pipes which do not flow full or which flow full without exerting pressure against the top of the pipe are classed as open channels and are treated in a separate chapter (Chapter X). A city water main carries water under pressure and is therefore an example of a pipe while a sewer which normally does not carry water under pressure is classed as an open channel.

Since friction losses in pipes are independent of pressure (Art. 96) the same laws apply to the flow of water both in pipes and open channels, and the formulas for each take the same general form. Some formulas are designed to be used either for pipes or open channels, but the more common practice is to use different formulas for the two classes of conduits.

89. Wetted Perimeter and Hydraulic Radius.—The wetted perimeter of any conduit is the line of intersection of its wetted surface with a cross-sectional plane. Thus for a pipe flowing full, d being the diameter, the wetted perimeter is equal to the circumference or πd, if flowing half full it is $\frac{1}{2}\pi d$.

The hydraulic radius of a conduit is the area of a cross-section of the stream divided by the wetted perimeter of that section. For a circular pipe flowing either full or half full the hydraulic radius, r, is evidently $d/4$ or $R/2$, R being the radius of the pipe.

The terms wetted perimeter and hydraulic radius are used more generally in connection with open channels than with pipes, but they are sometimes used in pipe formulas. Their application to open channels is discussed in Art. 109.

90. Critical Velocities in Pipes.—Under the conditions ordinarily encountered in hydraulic practice, water flows through pipes with a turbulent motion, that is, the water particles have a transverse as well as a longitudinal motion, and any particle near the center of the pipe at one time may be near its surface an instant later, occupying successively various transverse positions, while it is at the same time being propelled forward.

Though at any instant the water particles in a pipe where turbulent motion exists move forward with different velocities (Art. 91), the average longitudinal velocity of every particle is approximately the same. This may be shown by suddenly injecting a colored liquid into a pipe and observing the coloring matter where it discharges from the pipe. It will be observed that the coloring matter remains in a short prism even after the water has traveled a distance of 1000 diameters or more and that the water on either side of this prism is comparatively clear. This principle is made use of in measuring the velocity of flow through pipes.

At comparatively low velocities, water may be made to flow through small pipes without turbulence, that is with stream line motion. Under these conditions the water particles all flow in paths parallel to the axis of the pipe. The particles near the axis then flow with a higher velocity than the other particles, the velocities gradually becoming less as the distance from the center of the pipe increases, the lowest velocity being near the surface of the pipe. This retardation of velocities is caused by the viscosity of the water and friction between the moving water and the pipe.

The flow of water in small glass tubes has been studied experimentally by Reynolds[1] in the following manner. Water was drawn through the tubes from a glass tank in which the water had been allowed to come to rest, arrangements being made to introduce threads of colored water into the entrance of the tubes. Reynolds found, when the velocities were sufficiently low, that the streak of color extended as a beautiful straight line through the tube. As the velocity of the water was increased by small stages, a velocity was finally reached where the color suddenly

[1] Osborne Reynolds: *Phil. Trans.* Royal Society, 1882 and 1895.

mixed with the surrounding water. The velocity at which mixing began was evidently the velocity at which stream-line motion ended and turbulent motion began. It has been termed the *higher critical velocity.*

Reynolds also found that below a certain limiting velocity, when the water was disturbed it soon resumed stream-line motion but when the velocity was above this limit and the water was disturbed, even though stream-line motion had existed before the disturbance, turbulent motion occurred and stream-line motion could not be established. This limiting velocity is called the *lower critical velocity.*

The conditions of flow in a pipe $\frac{1}{2}$ in. in diameter are illustrated in Fig. 91. The line OA represents a gradual increase in velocity. If the water is not disturbed, stream-line motion will continue until a velocity somewhat greater than 3 ft. per second has been reached. Above this velocity the flow will always be turbulent. If now the water, starting with turbulent motion, is gradually decreased in velocity as indicated by the line AB, turbulent motion will continue until the velocity is reduced to about 0.5 ft. per second. Below this velocity stream-line motion will always exist.

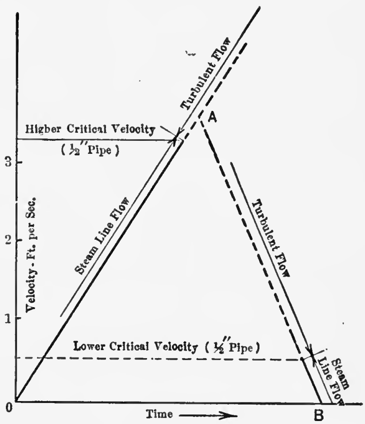

Fig. 91.

In general, it may be stated that for any pipe carrying water of a constant temperature:

(*a*) There is a certain velocity (the lower critical velocity) below which stream-line motion always exists.

(*b*) There is a certain velocity (the higher critical velocity) above which turbulent motion always exists.

(*c*) Between the lower critical velocity and the higher critical velocity the motion may be either stream-line or turbulent, depending upon the initial condition of flow.

Reynolds found that the critical velocity varied inversely as the diameter and directly as the viscosity of the water, the latter

being a function of the temperature, or expressed empirically, the lower critical velocity is

$$v_l = \frac{0.0388P}{d}, \qquad (1)$$

and the higher critical velocity is

$$v_h = \frac{0.246P}{d}, \qquad (2)$$

d being the diameter of the pipe in feet and

$$P = \frac{1}{1+0.034T+0.0002T^2}, \qquad (3)$$

being a viscosity coefficient in which T is the temperature in degrees Centigrade.

The following are critical velocities in feet per second obtained from the above formulas for pipes of different diameters at a temperature of 20° C. or 68° F.

	½ in.	1 in.	2 in.	4 in.	6 in.	12 in.
Lower......	0.53	0.26	0.13	0.07	0.04	0.02
Higher.....	3.35	1.68	0.84	0.42	0.28	0.14

As indicated by the above table the velocities entering into problems with which the engineer has to deal are ordinarily greater than the higher critical velocity. If not otherwise stated, therefore, turbulent flow will be assumed.

The laws governing stream-line motion are radically different from those governing turbulent motion.

91. Friction and Distribution of Velocities.—There is always friction between moving water and the surface of the conduit with which the water comes in contact. If this were not so the water in every part of the cross-section would flow with the same velocity. Fig. 92 shows the normal condition of flow in a straight pipe where there are no disturbing influences. Water particles adjacent to the surface are retarded by friction and viscosity (Art. 6) causes a retardation of the particles removed from the pipe surface. The maximum velocity is at the center, and lines of equal velocity are concentric rings as shown in cross-

section. The velocities in any longitudinal section when plotted as absicssas with the distance from one edge of the pipe as ordinates approximately define an ellipse. Experiments indicate that the mean velocity is about 0.85 of the maximum velocity. If d represents the diameter, the circle of mean velocity is approximately $0.13d$ from the surface of the pipe.

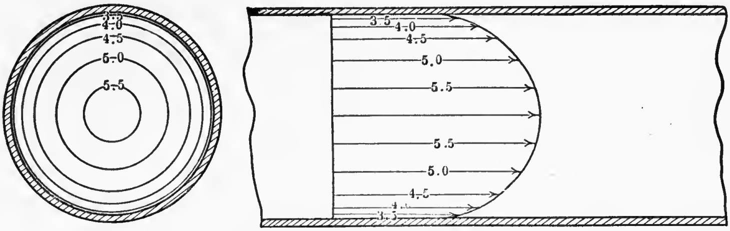

Fig. 92.—Distribution of velocities in straight pipe.

Any irregularity or obstruction in a pipe or any condition which causes the water to change its direction of flow will change the regular distribution of velocities. A bend in a pipe, for example, causes the line of maximum velocity to move from the axis of the pipe towards its concave side. Fig. 93 shows the

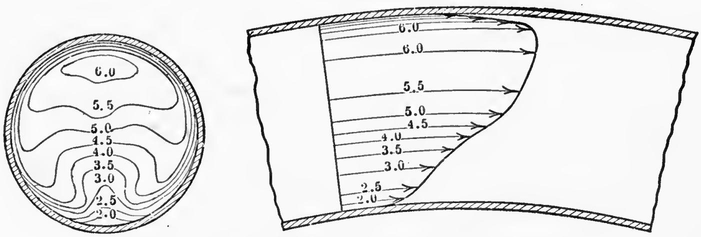

Fig. 93.—Distribution of velocities in curved pipe.

actual distribution of velocities in a curved pipe from measurements by Saph and Schoder.

92. Energy of Water in a Pipe.—The energy contained in a stream of water assumed to be moving with a uniform velocity, that is, with the same velocity in every part of its cross-section, is given by the formula,

$$KE = W\frac{v^2}{2g}, \quad \dots \dots \dots \quad (4)$$

W being the weight of water which moves past a cross-section in one second with a uniform velocity v. In the previous chapter, Art. 72, it has been shown that for the same average velocity, the energy of moving water in an open channel is greater for non-uniform velocity in a cross-section than for uniform velocity. The same is manifestly true for pipes, and since, as explained in the preceding article the velocity in pipes is never uniform, the kinetic energy of water in a pipe is given by the formula,

$$KE=\alpha W\frac{v^2}{2g}, \qquad \qquad (5)$$

in which α, a coefficient depending for its value upon the distribution of velocities in the pipe, is always greater than unity. Experiments by Bazin and others indicate that for a straight pipe, α has a mean value of about 1.06.

In problems involving the flow of water in pipes it is common to assume that the velocities at all points of a cross-section are equal, or that α equals unity and therefore, the kinetic energy contained in 1 lb. of water (or the velocity head) is equal to $\frac{v^2}{2g}$. Bernoulli's equation, when written between two points in a filament, then applies to the entire cross-section in which the points lie. The error introduced by assuming α equal to unity is not usually of serious consequence.

93. Continuity of Flow in Pipes.—In any pipe flowing full, within the limits of error resulting from the assumptions that water is incompressible and the pipe inelastic, at any given instant the same quantity of water is passing every cross-section of the pipe. This statement implies continuity of flow (see Art. 41) and holds true even when the flow is unsteady, a condition which exists when the head producing discharge is variable.

94. Loss of Head.—If there were no friction losses, the velocity at which water would discharge from a pipe, Figs. 94 and 95, would be $v_t=\sqrt{2gH}$, the same as for an orifice. For a horizontal pipe of uniform diameter, Fig. 94, there would be no pressure other than that resulting from the weight of water within the pipe and water would not rise in the piezometer tubes at m and n. In any long pipe or system of pipes, however, by

far the greater portion of the total head, H, is used in overcoming friction.

If there is no change in the diameter of a pipe, the difference in height of water columns in piezometer tubes at any two sections measures the loss of head due to friction between those sections. In Fig. 94 the loss of head between sections at m and n is h_m-h_n. In Fig. 95, which represents a system of pipes of different diameters, h_l-h_m is the loss of head between sections at l and m plus the increase in velocity head at m over that at l.

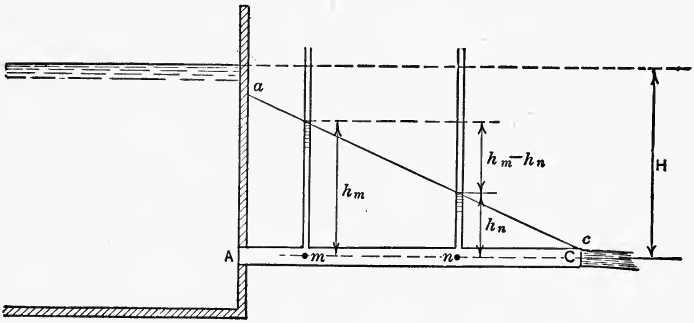

Fig. 94.—Pipe discharging from reservoir.

Similarly, h_m-h_n (Fig. 95) is the loss of head between sections at m and n minus the decrease in velocity head.

Considering the system of pipes illustrated in Fig. 95, Bernoulli's equation may be written between a point S in the water surface and another point E at the outlet as follows:

$$\frac{v_S^2}{2g}+\frac{p_S}{w}+Z_S=\frac{v_E^2}{2g}+\frac{p_E}{w}+Z_E+H_1, \quad . \quad . \quad . \quad . \qquad (6)$$

H_1 being the total loss of head from all causes and the remainder of the nomenclature being as indicated in the figure. Since v_S may be considered as equal to zero and $p_S=p_E=$atmospheric pressure, equation (6) reduces to

$$Z_S-Z_E=\frac{v_E^2}{2g}+H_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

or since $Z_S-Z_E=H$, the total head

$$H=\frac{v_E^2}{2g}+H_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

This means, that for a pipe discharging into the air, the total head is equal to the velocity head at the end of the pipe plus the sum of all friction losses. Since the velocity head at exit must be provided out of the total head, H, it is usually considered in the same manner as lost head. It should be remembered, however, that as the water leaves the pipe it still retains the energy represented by its velocity head.

In the case of a pipe connecting two reservoirs, Fig. 97, the water in the upper reservoir has a velocity of zero and it finally comes to rest in the lower reservoir. The reservoirs may be considered as parts of the pipe system in which the velocities

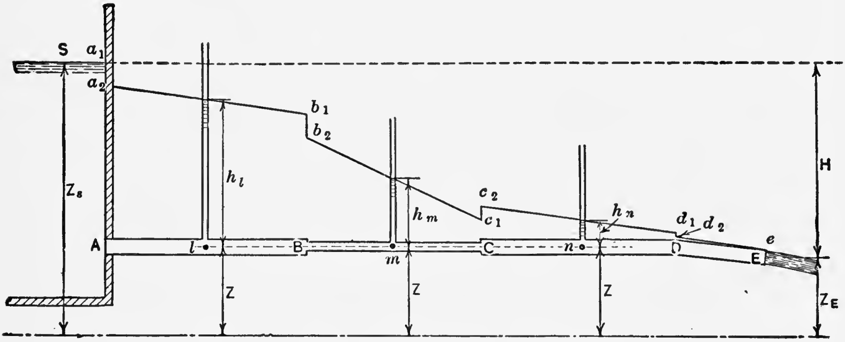

Fig. 95.—Pipe of more than one diameter.

are zero, the entire head, H, being utilized in overcoming friction; whence

$$H = H_1.$$

Frictional losses result from various causes. In any pipe in which the diameter remains unchanged and there are no conditions tending to disturb a regular distribution of velocities, the only loss of head is that due to the combined effects of viscosity and friction between the moving water and the surface of the pipe. This loss of head is commonly referred to as *loss of head due to friction*. Other losses of head are those which result from changing the velocity or direction of flow.

In ordinary pipe lines the loss of head due to friction is the greater portion of the total head. Frequently all other losses are so small in comparison as to be negligible. Cases arise, however, which require careful consideration of these losses and

serious errors may result from neglecting them. Losses of head other than the loss of head due to friction are commonly known as *minor losses.*

The following are the principal causes of loss of head in pipes, together with the symbols which will be used to designate these losses. (All losses except (*a*) are minor losses.)

(*a*) A continuous loss of head due to friction between the moving water and the inner surface of the pipe, and to viscosity. This loss is commonly referred to as the *loss of head due to friction,* and is designated by the symbol h_f.

(*b*) A loss of head at the entrance to a pipe, h_0, the loss occurring where the very low velocity in the reservoir (usually considered zero velocity) changes to the velocity in the pipe. This is called the *loss of head at entrance.*

(*c*) A loss of head, h_d, which occurs where a pipe discharges into a reservoir or other body of comparatively still water. This will be called the *loss of head at discharge.*

(*d*) A loss of head, h_c, at the place where a pipe changes to a smaller diameter thus causing an increase in velocity. This is called *loss of head due to sudden* or *gradual contraction,* depending upon whether the contraction takes place abruptly or by means of a tapered connection between the two pipes allowing the change in velocities to be made gradually. The loss of head at entrance (referred to under (*b*) above) is evidently a special case of loss due to contraction.

(*e*) A loss of head, h_e, at the place where a pipe changes to a larger diameter thus causing a decrease in velocity. This is called *loss of head due to sudden* or *gradual enlargement,* depending upon whether the enlargement takes place abruptly or by means of a tapered connection between the two pipes allowing the change in velocities to be made gradually. The loss of head at discharge (referred to under (*c*) above) is evidently a special case of loss of head due to enlargement.

(*f*) A loss of head, h_g, caused by obstructions in a pipe line, such as gates or valves. Obstructions cause the water to pass through a restricted area for a short distance, thus causing first a sudden increase in velocity and then a sudden return to the original velocity. This will be called the *loss of head due to obstructions.*

(*g*) A loss of head, h_b, at bends or curves in pipes, in addition

to the loss which occurs in an equal length of straight pipe. This is designated the *loss of head due to bends.*

If the symbol H_1 is used to designate all losses of head in a pipe line, the loss of head due to friction being represented by h_f and all minor losses by H_2,

$$H_1 = h_f + H_2, \quad \ldots \ldots \ldots \quad (9)$$

in which

$$H_2 = h_0 + h_d + h_c + h_e + h_g + h_b. \quad \ldots \quad (10)$$

95. Hydraulic Gradient.—The locus of the elevations to which water will rise in a series of piezometer tubes inserted in a pipe line is called the *hydraulic gradient* or *hydraulic grade line.* The hydraulic gradient of a straight pipe of uniform diameter having the same degree of roughness of interior surface throughout is a straight line. In Fig. 94 the line ac is the hydraulic gradient for the pipe. Where a pipe changes in diameter or where for any reason there is a change in velocity or direction of flow, there is a break in the hydraulic gradient, the change in elevation being the combined effects of the change in velocity head, where velocity changes occur, and the loss of head due to friction or turbulence. The broken line $a_1a_2b_1b_2c_1c_2d_1d_2e$, Fig. 95, represents the hydraulic gradient for the system of pipes shown. The hydraulic grade line thus indicates all losses of head and changes in velocity head.

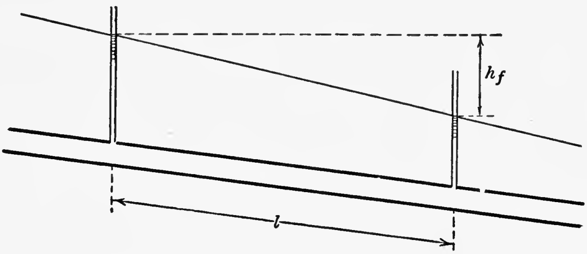

FIG. 96.

96. Loss of Head Due to Friction in Pipes.—Fig. 96 represents a straight pipe without obstructions or changes in diameter. The loss of head, h_f, in the length l is a measure of the resistance to flow. The laws governing this loss are intricate and are not subject to exact analysis. There are, however, certain general laws

which are in the nature of conclusions resulting from observation and experiment, which appear to govern fluid friction in pipes and which are expressed in the most generally accepted pipe formulas now in use. These laws may be briefly stated as follows:

(*a*) Frictional resistance is independent of the pressure within the pipe, and other things being equal:

(*b*) Friction between moving water and the inner surface of the pipe increases with the roughness of the surface. This may be expressed as a coefficient K whose value increases with the degree of roughness of the pipe.

(*c*) Friction between moving water and the inner surface of a pipe is directly proportional to the area of the wetted surface; that is, it is proportional to the product of the wetted perimeter and the length or πdl, d being the diameter and l the length of the pipe.

(*d*) As the cross-sectional area of the pipe increases, the retarding influence of viscosity becomes less, and it usually is considered to vary inversely as some power of the area, and therefore of the diameter or as $1/d^x$.

(*e*) Frictional resistance varies directly as some power of the velocity, or as v^n.

(*f*) Frictional resistance increases with the viscosity and therefore inversely with the temperature. This factor is usually omitted from pipe formulas, coefficients being selected which apply to average air temperatures.

Combining the factors expressed in (*b*), (*c*), (*d*) and (*e*) above, the total head lost is represented by the equation,

$$h_f = K \times \pi dl \times \frac{1}{d^x} \times v^n, \quad \ldots\ldots \quad (11)$$

or substituting K' and m for $K \times \pi$ and $x-1$ respectively, the general expression for loss of head due to friction in pipes may be written,

$$h_f = K' \frac{l}{d^m} v^n, \quad \ldots\ldots\ldots \quad (12)$$

since $\frac{h_f}{l} = s$, formula (12) may be transposed to the form

$$v = \left(\frac{1}{K'}\right)^{\frac{1}{n}} d^{\frac{m}{n}} s^{\frac{1}{n}},$$

or substituting K'' for $\left(\frac{1}{K'}\right)^{\frac{1}{n}}$, y for $\frac{m}{n}$, and z for $\frac{1}{n}$,

$$v=K''d^y s^z, \quad \ldots \ldots \ldots \quad (13)$$

or since the hydraulic radius r, for a circular pipe flowing full, equals $d/4$, or $d=4r$, formula (13) may be written,

$$v=K''\times 4^y\times r^y\times s^z,$$

or substituting K''' for $4^y K''$,

$$v=K'''r^y s^z. \quad \ldots \ldots \ldots \quad (14)$$

Each of the above formulas, though expressed differently, contains all of the factors, excepting temperature, which are believed to affect fluid friction. The base formulas for friction losses in pipes are commonly written in any of the three forms expressed by equations (12), (13), and (14).

The further consideration of loss of head due to friction in pipes must be purely empirical. The values of coefficients and exponents to be applied to the base formulas are determined from the available experimental data. Of the large number of published formulas for determining the loss of head due to friction in pipes, only a few are given.

It should be kept in mind that in all of the following formulas, h_f, l, d and other linear quantities must be expressed in feet and v must be expressed in feet per second.

97. The Chezy Formula.—This formula deserves a place of prominence among pipe formulas not only because it represents the first successful attempt to express friction losses in algebraic terms, but also because it embodies all of the laws of fluid friction as they are understood and applied at the present time, and with certain modifying factors that have been found necessary, its use is now more general than that of any other formula either for pipes or open channels.

As written by Chezy in 1775 this formula is

$$v=C\sqrt{rs}, \quad \ldots \ldots \ldots \quad (15)$$

in which v is the mean velocity, r is the hydraulic radius and $s=\frac{h_f}{l}$ is the rate of slope of the hydraulic gradient. It will be observed

that formula (15) is of the same form as (14), y and z being each $\frac{1}{2}$ and C being substituted for K'''.

The coefficient C was supposed by Chezy to be constant, but it is now known to vary with the degree of roughness of the surface with which the water comes in contact as well as with the velocity and hydraulic radius (or diameter). Since C appears to be a function of v and r the Chezy formula evidently does not accurately express the law of fluid friction. In the ideal formula, the coefficient would vary only with the roughness of the channel, and many attempts have been made to obtain a formula with such a coefficient expressing v as a function of r and s. These attempts have met with rather indifferent success.

Formula (15) is used with an accompanying table giving values of C for different velocities, diameters and kinds of pipe. The table on page 148 gives approximate average values of C for four different kinds of pipe, as obtained from the available experimental data.

In an account of experiments on the flow of water in pipes, published by Darcy[1] in 1857, he expressed the Chezy formula in the form,

$$h_f=f\frac{l}{d}\frac{v^2}{2g}, \qquad (16)$$

the relations between C and f in formulas (15) and (16) being

$$f=\frac{8g}{C^2} \quad \text{and} \quad C=2\sqrt{\frac{2g}{f}}.$$

It will be observed that formula (16) may be obtained from formula (12) by writing $n=2$ and $m=1$, the two formulas being of the same general form.

From his experiments Darcy deduced the following values of f, as representing the mean of his observations.

For new, clean cast-iron pipes, d being the diameter of the pipe in feet,

$$f=0.02+\frac{0.02}{12d}.$$

For old cast-iron pipes,

$$f=0.04+\frac{0.04}{12d}.$$

[1] M. H. Darcy: Recherches Expérimentales Relatives au Mouvement de l'eau dans les Tuyaux. Paris, 1857.

VALUES OF C IN THE CHEZY FORMULA, $v = C\sqrt{rs}$

Diameter of Pipe, Inches	Clean Cast-iron Pipe Velocity, Feet per Second				Old Cast-iron Pipe Velocity, Feet per Second				Clean Wooden Pipe, Velocity, Feet per Second				Clean Concrete Pipe Velocity, Feet per Second			
	2	5	10	20	2	5	10	20	2	5	10	20	2	5	10	20
1	81	85	89	92	62	62	62	62	75	84	91	97	69	75	78	82
2	88	93	98	100	67	67	67	67	82	91	99	105	75	82	85	90
3	93	98	103	106	71	71	71	71	86	96	104	111	79	86	89	94
4	96	101	106	110	73	73	73	73	89	99	108	115	82	89	92	98
5	99	104	109	113	75	75	75	75	92	102	111	118	84	92	95	100
6	101	106	112	116	77	77	77	77	94	105	114	121	86	94	97	103
8	105	110	116	120	80	80	80	80	97	108	118	125	89	97	101	106
10	108	113	119	123	82	82	82	82	100	111	121	129	92	100	104	110
12	110	116	122	126	84	84	84	84	102	114	124	132	94	102	106	112
15	113	119	126	130	86	86	86	86	105	117	128	136	97	105	109	115
18	116	122	128	133	88	88	88	88	107	120	131	139	99	107	112	118
24	120	126	133	137	91	91	91	91	111	124	135	144	103	111	116	122
30	123	130	137	141	94	94	94	94	114	128	139	148	105	114	119	126
36	126	133	140	144	96	96	96	96	117	131	142	151	108	117	122	129
42	129	136	143	147	98	98	98	98	119	133	145	154	110	119	124	131
48	131	138	145	150	100	100	100	100	121	136	147	157	112	121	126	133
60	134	142	149	154	103	103	103	103	125	139	152	161	115	125	130	137
72	137	145	153	158	105	105	105	105	128	142	155	165	118	128	133	140
84	140	148	156	161	107	107	107	107	130	145	158	168	120	130	135	143
96	143	150	158	163	109	109	109	109	132	148	161	171	122	132	138	145

VALUES OF f IN THE CHEZY FORMULA, $h_f = f \frac{l}{d} \frac{v^2}{2g}$

Diameter of Pipe, Inches	Clean Cast-iron Pipe Velocity, Feet per Second				Old Cast-iron Pipe Velocity, Feet per Second				Clean Wooden Pipe Velocity, Feet per Second				Clean Concrete Pipe Velocity, Feet per Second			
	2	5	10	20	2	5	10	20	2	5	10	20	2	5	10	20
1	.039	.035	.034	.030	.071	.071	.071	.071	.047	.037	.032	.028	.054	.047	.043	.039
2	.033	.030	.028	.025	.059	.059	.059	.059	.039	.031	.027	.023	.045	.039	.036	.033
3	.030	.027	.025	.023	.054	.054	.054	.054	.035	.028	.024	.021	.041	.035	.033	.030
4	.028	.025	.024	.021	.050	.050	.050	.050	.033	.026	.022	.020	.038	.033	.030	.028
5	.026	.024	.022	.020	.047	.047	.047	.047	.031	.025	.021	.019	.036	.031	.029	.026
6	.025	.023	.021	.019	.045	.045	.045	.045	.030	.024	.020	.018	.034	.030	.027	.025
8	.023	.021	.020	.018	.042	.042	.042	.042	.028	.022	.019	.017	.032	.028	.025	.023
10	.022	.020	.019	.017	.040	.040	.040	.040	.026	.021	.018	.016	.030	.026	.024	.022
12	.021	.019	.018	.016	.038	.038	.038	.038	.025	.020	.017	.015	.029	.025	.023	.021
15	.020	.018	.017	.015	.036	.036	.036	.036	.024	.019	.016	.014	.027	.024	.022	.020
18	.019	.017	.016	.014	.034	.034	.034	.034	.023	.018	.015	.014	.026	.023	.021	.019
24	.018	.016	.015	.013	.032	.032	.032	.032	.021	.017	.014	.013	.024	.021	.019	.018
30	.017	.015	.014	.013	.030	.030	.030	.030	.020	.016	.014	.012	.023	.020	.018	.017
36	.016	.014	.014	.012	.029	.029	.029	.029	.019	.015	.013	.011	.022	.019	.017	.016
42	.015	.014	.013	.012	.027	.027	.027	.027	.018	.014	.012	.011	.021	.018	.017	.015
48	.015	.013	.013	.011	.026	.026	.026	.026	.018	.014	.012	.011	.020	.018	.016	.015
60	.014	.013	.012	.011	.025	.025	.025	.025	.017	.013	.011	.010	.019	.017	.015	.014
72	.013	.012	.011	.010	.024	.024	.024	.024	.016	.013	.011	.010	.019	.016	.015	.013
84	.013	.012	.011	.010	.023	.023	.023	.023	.015	.012	.010	.009	.018	.015	.014	.013
96	.012	.011	.011	.010	.023	.023	.023	.023	.015	.012	.010	.009	.017	.015	.014	.012

Later experiments have indicated that f (similar to C in formula (15)) varies with v as well as with d. Formula (16), the same as formula (15), is used more satisfactorily with an accompanying table of coefficients. The table on page 149 gives average values of f for different kinds of pipe. The two formulas, (15) and (16), used in connection with the coefficients tabulated on pages 148 and 149, respectively, will give the same results. The formula is written in the two forms merely as a convenience in solving different types of problems.

The values of C and f given in the tables on pages 148 and 149 refer not only to pipes of the particular materials listed but to any pipes of similar degrees of roughness. The problem of selecting the proper coefficient for a given condition is one with which the engineer is continually confronted, and in making such a selection experience is the best teacher. It is important to know the most probable value of a coefficient and the maximum per cent of error likely to result from its use. The average values of C and f listed in the tables may give results in error as much as 20 per cent plus or minus.

The following values of f for $2\frac{1}{2}$-in. fire hose are given by Freeman:

VALUES OF f IN CHEZY FORMULA FOR $2\frac{1}{2}$-IN. FIRE HOSE

Description.	Velocity in Feet per Second.				
	4	6	10	15	20
Unlined canvas	.038	.038	.037	.035	.034
Rough rubber-lined cotton	.032	.031	.031	.030	.029
Smooth rubber-lined cotton	.024	.023	.022	.019	.018

98. Hazen-Williams Formula.—This formula which is of the form of (14) is

$$v = C_1 r^{0.63} s^{0.54} 0.001^{-0.04}. \quad \ldots \ldots \quad (17)$$

It is designed for both pipes and open channels, but is used more commonly in connection with pipes. The selection of exponents was made with a view to obtaining a minimum variation in C_1 for all conduits of the same degree of roughness. In other

words the aim was to select values for exponents such that C_1 would be, as nearly as practicable, a function only of the degree of roughness of the channel and not of r and s. The following is written by the authors [1] of the formula.

"If exponents could be selected agreeing perfectly with the facts, the value of C_1 would depend upon the roughness only, and for any given degree of roughness C_1 would then be a constant. It is not possible to reach this actually, because the values of the exponents vary with different surfaces, and also their values may not be exactly the same for large diameters and for small ones, nor for steep slopes and for flat ones. Exponents can be selected, however, representing approximately average conditions, so that the value of C_1 for a given condition of surface will vary so little as to be practically constant. Several such 'exponential' formulas have been suggested. These formulas are among the most satisfactory yet devised, but their use has been limited by the difficulty in making computations by them. This difficulty was eliminated by the use of a slide-rule constructed for that purpose.

"The exponents in the formula used were selected as representing as nearly as possible average conditions, as deduced from the best available records of experiments upon the flow of water in such pipes and channels as most frequently occur in waterworks practice. The last term, $0.001^{-0.04}$, is a constant, and is introduced simply to equalize the value of C_1 with the value in the Chezy formula, and other exponential formulas which may be used, at a slope of 0.001 instead of at a slope of 1."

Since $0.001^{-0.04} = 1.318$, the formula may be written,

$$v = 1.318 C_1 r^{0.63} s^{0.54}. \qquad (18)$$

The authors of the formula give the following values of C_1 for pipes:

For extremely smooth and straight pipes.... $C_1 = 140$
For very smooth pipes.................... $C_1 = 130$
For new riveted steel pipes................ $C_1 = 110$

For estimating discharges of pipe lines where the carrying capacity after a series of years is the controlling factor, values of

[1] WILLIAMS AND HAZEN: Hydraulic Tables. Third Edition, 1920.

$C_1=100$ for cast-iron pipe and $C_1=95$ for riveted steel are recommended. For the smaller sizes of pipes a somewhat lower value of C_1 should be used.

For smooth wooden pipes or wooden-stave pipes, $C_1=120$.

For vitrified pipes, $C_1=110$.

For old iron pipes in bad condition, $C_1=80$ to 60, and for small pipes badly tuberculated, C_1 may be as low as 40.

99. King Formula.—Consider the base formula (12), which for convenience of reference is here repeated,

$$h_f=K'\frac{l}{d^m}v^n. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

It has been found from experiments on a great many kinds and sizes of pipes that no value of n can be found which does not vary under different conditions of flow. The extreme range of variation, from investigations by Lea [1] and others, is from about 1.75 to 2.08. On the other hand, it appears that a mean value of m of 1.25 may be assumed without introducing any serious inconsistencies. Formula (12) has therefore been modified by Lea to the form,

$$h_f=K_1\frac{l}{d^{1.25}}v^n, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

K_1 and n each being given variable values depending upon the degree of roughness of the pipe.

The formula also may be written,

$$h_f=K\frac{l}{d^{1.25}}\frac{v^2}{2g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

in which

$$K=\frac{2gK_1}{v^{2-n}}.$$

Formula (20) expresses the loss of head due to friction as a function of the velocity head. This is sometimes convenient since miner losses are usually thus expressed (Art. 102). The formula is somewhat simpler to use than a formula in which v has a fractional exponent. On the other hand, since K varies

[1] F. C. Lea: Hydraulics, p. 139. H. W. King: Handbook of Hydraulics, p. 159.

only with v and not with d, a much simpler table of coefficients is required for formula (20) than is required for the Chezy formula.

Average values of K to be used with formula (20) are given in the table below. These values are the same as the values of f on page 149 for $d=12$ inches.

VALUES OF K IN THE FORMULA $h_f=K\frac{l}{d^{1.25}}\frac{v^2}{2g}$

Velocity, in feet per second	Clean cast-iron pipe	Old cast-iron pipe	Clean wooden pipe	Clean concrete pipe
2	.021	.038	.025	.029
5	.019	.038	.020	.025
10	.018	.038	.017	.023
20	.016	.038	.015	.021

Formula (20) may also be transposed and written,

$$v=\sqrt{\frac{2g}{K}\frac{h_f}{l}}\,d^{0.625},$$

in which $h_f/l=s$ is the slope of the hydraulic gradient. Substituting this value and writing C' for $\sqrt{\frac{2g}{K}}$,

$$v=C's^{1/2}d^{0.625}. \qquad (21)$$

In this form the formula is more convenient for certain types of problems. Average values of C' are given in the following table:

VALUES OF C' IN THE FORMULA $v=C's^{1/2}d^{0.625}$

Velocity, in feet per second	Clean cast-iron pipe	Old cast-iron pipe	Clean wooden pipe	Clean concrete pipe
2	55	42	51	47
5	58	42	57	51
10	61	42	62	53
20	63	42	66	56;

Formulas (15), (16), (20) and (21), when used with their accompanying tables of coefficients, all give the same results.

100. General Discussion of Pipe Formulas.—The foregoing formulas represent the more common types of formulas for determining the loss of head due to friction in pipes. There are an indefinite number of formulas, many of which possess merit. The choice of one formula over another is not of so great importance as the careful and intelligent use of the formula after it is selected. The engineer should select the formula for general use which he believes to be in the most convenient form, and after adopting it he should endeavor to become familiar with its coefficients. The tables contained in this volume are sufficient for class room exercises, but the practicing engineer should extend his knowledge of coefficients by study and observation, and obtain values from actual measurements whenever the opportunity offers.

101. Friction Formula for Non-turbulent Flow.—If v' is the velocity in feet per second, d_i the diameter of the pipe in inches, h_f the friction loss in a length l, and P the viscosity coefficient (formula (3), Art. 90), the velocity in a pipe where stream-line flow exists according to Reynolds is

$$v' = \frac{361 d_i^2 h_f}{lP}. \quad \quad (22)$$

If a case be assumed where $d_i = 1$ in., $h_f = 1$ ft. and $l = 100$ ft., v' for a temperature of zero degrees Centigrade is 3.61 ft. per second, and for higher temperatures the velocity would be greater. The table on page 138 shows this velocity to be above the higher critical velocity and the flow must be turbulent. Formula (22) therefore does not apply and one of the formulas for turbulent flow should be used.

102. Detailed Study of Hydraulic Gradient and Minor Losses.—In the discussion of loss of head due to friction (Art. 96), it has been shown that, other things being equal, the loss of head varies as v^n and that usually n is less than 2 but does not vary greatly from this value. In some formulas, therefore, this loss of head is expressed as a function of the velocity head and coefficients varying in value with v are applied to the formulas to make them represent average friction losses as given by experiments.

In a similar manner it has been found that minor losses (Art.

94) vary roughly with the square of the velocity and they are commonly expressed in formulas as functions of the velocity head. Variable coefficients are then applied to these formulas so as to make them give losses in accordance with the available experimental data. These losses (see Art. 94) expressed algebraically are

$$h_0=K_0\frac{v^2}{2g},\quad h_a=K_a\frac{v^2}{2g},\quad h_c=K_c\frac{v^2}{2g},\quad \text{etc.,}$$

and formula (10), page 144, may be written,

$$H_2=K_0\frac{v^2}{2g}+K_a\frac{v^2}{2g}+K_c\frac{v^2}{2g}+K_e\frac{v^2}{2g}+K_g\frac{v^2}{2g}+K_b\frac{v^2}{2g}. \quad . \quad (23)$$

In the above equation, v is a general expression for velocity. It is the velocity in the pipe where the loss of head occurs and in case of enlargement or contraction it is the velocity in the smaller pipe. K_0, K_a, K_c, etc., are variable coefficients whose values must be determined from experiments.

Fig. 97 illustrates *entrance* and *discharge* conditions for a pipe leading from one reservoir into another reservoir at a lower elevation. The water starts with zero velocity in the upper reservoir, finally coming to rest in the lower reservoir, and all of the energy represented by the difference in elevation of water surfaces is utilized in overcoming resistance.

In Fig. 97, a_1a_2 represents the hydraulic grade line which results from changing the velocity of the water from zero to the velocity which it attains in the pipe. The vertical distance between a_1 and a_2, that is, the distance which a_2 is below the surface of the water, is the velocity head or $v^2/2g$, where v is the mean velocity in the pipe. The line a_1a_2 must be considered as the hydraulic gradient of some particular filament of water, such as xy, since points in other filaments which are the same horizontal distance from the entrance to the pipe may have different velocities and therefore different hydraulic gradients. It may appear that the pressure at any point in the filament should be that due to the weight of the water column above it. This would be true if the laws of hydrostatics might be applied. The laws of hydrostatics do not, however, apply to water in motion, the pressure being less than it would be at the same depth for water at rest. That this is true has been proved experimentally. It also follows from writing Bernoulli's equation between a point x where the

velocity is practically zero and a point y at the entrance to the pipe where the velocity equals v, the velocity in the pipe. Assuming the points to be of the same elevation the equation becomes

$$h_x + \frac{v_x^2}{2g} = h_y + \frac{v^2}{2g},$$

or since v_x is practically zero,

$$h_y = h_x - \frac{v^2}{2g}. \qquad (24)$$

The head lost at entrance to a pipe takes place within a distance of about two or three diameters from the entrance and is similar to the loss of head in a short tube. The line a_2a_3, Fig. 97, is the portion of the hydraulic gradient which shows this loss of

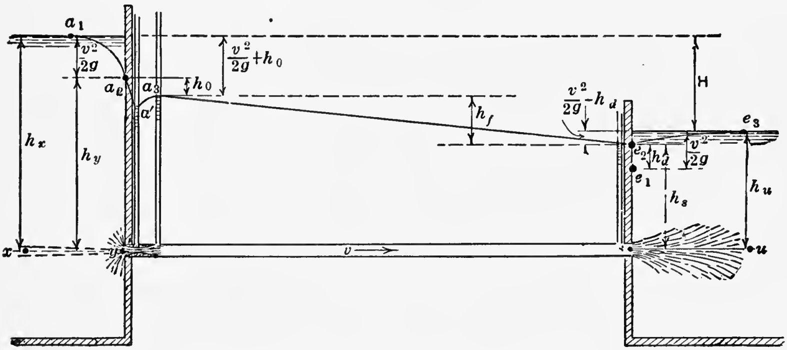

Fig. 97.—Pipe connecting two reservoirs.

head. There is a depression in the hydraulic gradient at a' because of the contraction of the jet. Vertically below a_3, the jet has expanded and fills the tube. The head lost at entrance is the vertical distance between a_2 and a_3, or h_0.

Since the first two or three diameters of a pipe are similar to a short tube, entrance losses for pipes may be considered to be the same as for short tubes. The general formula for loss of head at entrance to a pipe is then (formula (22), page 81),

$$h_0 = \left(\frac{1}{C^2} - 1\right)\frac{v^2}{2g} = K_0\frac{v^2}{2g}, \qquad (25)$$

in which the coefficient of discharge, C, depends for its value upon the conditions at entrance, and $K_0 = \frac{1}{C^2} - 1$. For convenience of

reference, values of C and K_0 given in Chapter VII, are repeated in the following table:

COEFFICIENTS FOR DETERMINING LOSS OF HEAD AT ENTRANCE TO PIPES

Entrance to pipe	Reference	C	K_0
Inward projecting..............	Art. 63	0.75	0.78
Sharp cornered.................	Art. 58	0.82	0.50
Slightly rounded...............	Art. 57	0.90	0.23
Bell mouth.....................	Art. 57	0.98	0.04

Since the effect of entrance conditions can not be determined accurately the selection of a proper value of K_0 is to some extent a matter of judgment. Unless the entrance is known to be other than sharp cornered, a value of 0.5 is commonly used.

Conditions at the outlet of a pipe may be illustrated in a similar manner. If there were no loss of head where water enters the lower reservoir the hydraulic grade line would connect a_3 and e_1 (Fig. 97), the latter point being $v^2/2g$ below the surface of the water. The distance e_1e_2 represents the portion of the velocity head lost through shock and turbulence. This may be illustrated by writing Bernoulli's equation between a point s at the outlet of the pipe where the velocity equals v, the velocity in the pipe, and a point u where the velocity v_u is practically zero. h_d equals the loss of head due to turbulence. If the two points are at the same elevation,

$$h_s+\frac{v^2}{2g}=h_u+\frac{v_u^2}{2g}+h_d, \quad \ldots\ldots \quad (26)$$

or since $v_u=0$,

$$h_s=h_u-\frac{v^2}{2g}+h_d, \quad \ldots\ldots\ldots \quad (27)$$

h_u-h_s represents the portion of the velocity head which is not lost but which is reconverted into pressure head. The rate at which this reconversion takes place is represented in the figure by the line e_2e_3, the end of the hydraulic gradient.

Expressing the head lost at discharge, h_d, as a function of the velocity head,

$$h_d=K_d\frac{v^2}{2g}. \quad \ldots\ldots\ldots\ldots \quad (28)$$

The loss at discharge is the special case of loss of head due to sudden enlargement in which the ratio of smaller to larger diameter is practically zero. Values of K_d may therefore be taken from column 2 of the table on page 161. Since these values are nearly unity for the ordinary velocities encountered in pipes it is commonly considered that the entire velocity head is lost.

Change in gradient resulting from sudden contraction in pipes is illustrated in Fig. 98. If there were no loss of head between any two points, on opposite sides of the contracted section, the difference in heights of water columns in two piezometer tubes as *b* and *e*, above these points would measure the gain in velocity head. If the two points are considered so close together that pipe friction may be neglected, the difference in height of water

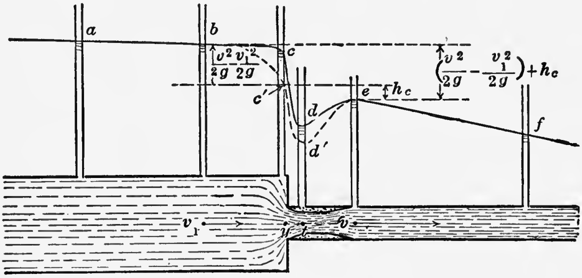

Fig. 98.—Sudden contraction in pipe.

columns *b* and *e* measures the gain in velocity head plus the loss of head due to sudden contraction.

The hydraulic gradient as determined experimentally is illustrated by the line *abcdef*. There is a depression at *d*, due to contraction of the jet, similar to the depression at *a'* in the hydraulic grade line of Fig. 97. The piezometer tube *c* measures the pressure in the corner where there is little or no velocity. If piezometer tubes, *c* and *d*, were arranged to measure pressures near the axis of the pipe where the velocities are higher, the hydraulic gradient would be below *bcde* and would resemble *bc'd'e*.

It is important to note that the ordinary piezometer tube, which is set flush with the inner surface of the pipe, measures the pressure at the surface of the pipe but does not necessarily measure the pressure at points in the same cross-section at some

distance from the surface. In smooth straight pipes the difference between pressures at the surface and interior points is probably not great but the difference may be quite large near sections where changes in diameter occur.

The loss of head due to sudden contraction expressed as a function of the velocity head is

$$h_c = K_c \frac{v^2}{2g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

in which K_c is an empirical coefficient, and v is the velocity in the smaller pipe. The following table gives experimental values of K_c.

VALUES OF THE COEFFICENT K_c, FOR SUDDEN CONTRACTION

Velocity in smaller pipe, v	Ratio of Smaller to Larger Diameter									
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
2	0.49	0.49	0.48	0.45	0.42	0.38	0.28	0.18	0.07	0.03
5	.48	.48	.47	.44	.41	.37	.28	.18	.09	.04
10	.47	.46	.45	.43	.40	.36	.28	.18	.10	.04
20	.44	.43	.42	.40	.37	.33	.27	.19	.11	.05
40	.38	.36	.35	.33	.31	.29	.25	.20	.13	.06

The loss of head at entrance to pipes is a special case of loss of head due to contraction. If the body of water is large the conditions conform approximately to a ratio of diameters of zero, and for a square-cornered entrance, where the end of the pipe is flush with a wall having a plane surface, the values of K_0 are comparable with the values of K_c in the second column of the above table.

If the change to a smaller diameter takes place gradually, as is the case with a gradually tapering section connecting the two pipes, or if the corners of the smaller pipe are rounded so as to reduce contractions, values of K_c will be much smaller than those given for a sudden reduction of diameter. If the change is made as gradually as in a Venturi meter or if a bell-mouth connection

(page 85) between the two pipes is used, K_c may become practically negligible.

Change in gradient resulting from sudden enlargement is shown in Fig. 99. In this case the change in height of water columns in piezometer tubes *b* and *e*, before and after enlargement, measures the gain in pressure head and this gain is equal to the loss in velocity head minus the loss of head due to sudden enlargement, the loss of head due to pipe friction being assumed so small as to be negligible. This may be verified by writing Bernoulli's equation between points on either side of the enlargement.

The hydraulic gradient as plotted from experiments by Gibson is *abcde*. The pressures shown by tubes *c* and *d* being

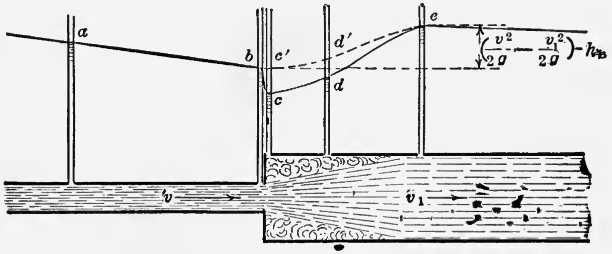

Fig. 99.—Sudden enlargement in pipe.

measured at the surface of the pipe are undoubtedly less than if they were measured near the center of the pipe. In this case the piezometer tube *c* would read practically the same as the tube *b* and the pressure would be greater than that indicated. The portion of the hydraulic gradient *bcde* would then be similar to *bc′d′e*.

The loss of head due to sudden enlargement expressed as a function of the velocity head is

$$h_e = K_e \frac{v^2}{2g}. \qquad (30)$$

Archer[1] has shown from an investigation of his own experi-

[1] W. H. Archer: Loss of Head Due to Enlargements in Pipes. *Trans. Amer. Soc. Civ. Eng.*, vol. 76, pp. 999–1026 (1913).

ments and the experiments of others that h_e is quite accurately represented by the formula,

$$h_e = 1.098\frac{(v-v_1)^{1.919}}{2g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

in which v is the velocity in the smaller pipe and v_1 is the velocity in the larger pipe. Experiments at the University of Michigan indicate that Archer's formula holds quite accurately in the limit where v_1 is zero. In this case the conditions become those of loss of head at discharge, discussed on page 157.

By equating the values of h_e given by equations (30) and (31) and transposing, the following value of K_e is obtained:

$$K_e = \frac{1.098}{v^{0.081}}\left(1 - \frac{d^2}{d_1^2}\right)^{1.919}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

d/d_1 being the ratio of the smaller to the larger diameter. The following table of values of K_e are computed from formula (32):

Values of the Coeffcient K_e, for Sudden Enlargement

Velocity in smaller pipe, v	Ratio of Smaller to Larger Diameter									
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
2	1.00	1.00	0.96	0.86	0.74	0.60	0.44	0.29	0.15	0.04
5	.96	.95	.89	.80	.69	.56	.41	.27	.14	.04
10	.93	.91	.86	.77	.67	.54	.40	.26	.13	.04
20	.86	.84	.80	.72	.62	.50	.37	.24	.12	.04
40	.81	.80	.75	.68	.58	.47	.35	.22	.11	.03

The loss of head due to enlargement in pipes may be reduced by changing the diameters gradually. If the diameter increases at a uniform rate, the amount of loss increases as the angle between the axis and surface of the pipe increases and is practically negligible for very small angles. Experimental values of K_e have not been well determined for gradual enlargements, but those given in the following table are the approximate mean of such data as are available. There are not sufficient experimental data to determine the extent to which K_e varies with the velocity.

VALUES OF THE COEFFCIENT K_e, FOR GRADUAL ENLARGEMENT

Angle between axis and surface of pipe	Ratio of Smaller to Larger Diameter								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
5°	0.04	0.04	0.04	0.04	0.04	0.04	0.03	0.02	0.01
15°	.16	.16	.16	.16	.16	.15	.13	.10	.06
30°	.49	.49	.48	.48	.46	.43	.37	.27	.16
45°	.64	.63	.63	.62	.60	.55	.49	.38	.20
60°	.72	.72	.71	.70	.67	.62	.54	.43	.24

Gates or Valves when partially closed obstruct the flow and cause a loss of head and consequent drop in the hydraulic gradient. The difference in elevation of the hydraulic gradient on opposite sides of the obstruction measures the loss of head due to the obstruction. Following the form used for other losses, the loss of head in pipes due to gates, valves, or other obstructions may be written,

$$h_g = K_g \frac{v^2}{2g}, \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

v being the mean velocity in the pipe. Experiments indicate that K_g does not vary appreciably with the velocity but increases with the amount of restriction. The following values of K_g are the average values obtained from the best experimental data available.

VALUES OF THE COEFFICIENT K_g, FOR OBSTRUCTIONS IN PIPES

Ratio of area of opening to cross-sectional area of pipe	K_g	Ratio of area of opening to cross-sectional area of pipe	K_g
0.1	13.5	0.5	2.7
0.15	10.0	0.6	1.8
0.2	8.0	0.7	1.1
0.3	5.7	0.8	0.6
0.4	4.0	0.9	0.2

Bends or curves in pipes cause a gradual drop in the hydraulic gradient which is in excess of the drop that would occur in an

equal length of the same kind of straight pipe. From a careful investigation of the available experimental data Fuller[1] deduced the following empirical formula,

$$h_b = cv^{2.25}, \qquad (34)$$

h_b being the loss of head due to bends in excess of the loss which would occur in an equal length of straight pipe, v being the mean velocity in the pipe and c being a coefficient whose value varies with the radius, R, of the axis of the pipe expressed in feet. Fuller gives the following values of c for bends of 90°: For $R=0$, $c=0.0135$; for $R=0.5$, $c=0.0040$; for $R=1$, $c=0.00275$; for $R=3$, $c=0.0024$; for $R=6$, $c=0.0023$; for $R=10$, $c=0.00335$; for $R=20$, $c=0.0060$; for $R=30$, $c=0.0070$; for $R=40$, $c=0.0075$; for $R=60$, $c=0.0086$.

Following the form used for other losses,

$$h_b = K_b \frac{v^2}{2g}, \qquad (35)$$

and equating the right-hand members of equations (34) and (35) and reducing,

$$K_b = 2gcv^{0.25}. \qquad (36)$$

From this formula the values of K_b given in the following table have been computed:

VALUES OF THE COEFFICIENT, K_b, FOR LOSS OF HEAD DUE TO BENDS OF 90°

Mean velocity in pipe, v	Radius of Bend in Feet										
	0	0.5	1	2	6	8	10	20	30	40	50
2	1.03	0.31	0.21	0.19	0.18	0.21	0.26	0.45	0.53	0.57	0.61
5	1.30	.38	.26	.23	.22	.26	.32	.57	.67	.72	.77
10	1.54	.46	.31	.28	.26	.31	.38	.68	.79	.86	.92
20	1.84	.54	.37	.33	.31	.37	.46	.81	.95	1.02	1.08
40	2.18	.65	.44	.39	.37	.44	.54	.97	1.12	1.21	1.30

From the above table it will be seen that the minimum loss of head from bends occurs when the radius of the axis of the

[1] W. E. FULLER: Loss of Head in Bends. *Journal of New England Water Works Association,* December, 1913.

pipe is from 2 to 6 ft. For bends of 45° the coefficients will be about three-fourths and for $22\frac{1}{2}°$ about one-half of the values given in the table.

103. Part of Pipe Above Hydraulic Gradient.—Fig. 100 shows a pipe of uniform diameter leading from a reservoir and discharging under the head H. The summit, M, is a distance y above the straight line BeC but at a lower elevation than the water surface in the reservoir. Two conditions will be considered: first, where $y<\frac{p_a-p_v}{w}$, and second, where $y>\frac{p_a-p_v}{w}$, p_a being atmospheric pressure and p_v being the vapor pressure corresponding to the temperature of the water in the pipe.

Assume the pipe $AMSC$, Fig. 100, to be empty when water

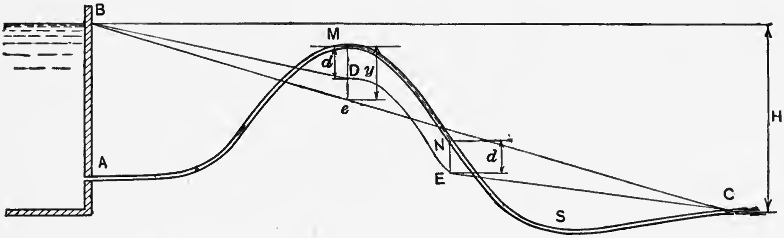

Fig. 100.—Part of pipe above hydraulic gradient.

is turned into it at A. Water will first rise to the summit M and begin to flow down the decline toward the depression S, at which point it will collect and seal the pipe entrapping air between M and S. Eventually water will discharge from the outlet C. If the velocity is high enough the air entrapped between M and S will be removed by the flowing water, otherwise it will remain there and obstruct the flow. In such cases the air may be removed by a suction pump at the summit. If there is no air in the pipe and $y<\frac{p_a-p_v}{w}$, assuming the loss of head to be uniform, the hydraulic gradient will be the straight line BeC and the flow will be the same as though all of the pipe were below the hydraulic gradient.

If $y>\frac{p_a-p_v}{w}$, the flow of water will be restricted, even though all air is exhausted from the pipe, and the hydraulic gradient

will no longer be a straight line. This condition is illustrated in Fig. 100. The hydraulic gradient is a straight line to a point, D, which is a distance $d=\frac{p_a-p_v}{w}$ below the summit, M. At M (assuming no air in the pipe) the absolute pressure in the pipe is the vapor pressure corresponding to the temperature within the pipe and this pressure continues on down to N, the pipe flowing partially full between M and N. Throughout all portions of the pipe flowing full the velocity must necessarily be the same since the discharge is constant and therefore, assuming a uniform degree of roughness for the pipe, the slope of hydraulic gradient in such portions must be uniform. In other words, the slope of EC must be the same as the slope of BD. Throughout the distance where the pipe is not flowing full, the hydraulic gradient, represented by the line DE, is the same vertical distance, d, below the water surface in the pipe. The point E is the intersection of the line CE, parallel to BD, and the line DE. The section at N where the pipe begins to flow full is vertically above E.

The conditions of flow, especially at low velocities, are not usually as favorable as those described above, because of the tendency of air to collect at a summit. Water flowing at low velocities will not remove air and may even liberate it, and cause air to collect at high places such as M, Fig. 100. The condition will be worse at summits above the hydraulic gradient if the pipe leaks, since the movement of air will be inward. In such cases the occasional operation of an air pump at the summit will be necessary to remove the air. At a summit below the hydraulic gradient, where the pressure within the pipe is greater than atmospheric, the air which collects may be removed through a valve.

Air at a summit which is below the elevation of the water surface will not stop the flow of water entirely but will cause a portion of the pipe to flow partially full. Summits in pipe lines are always objectionable, and especially so are summits above the hydraulic gradient. Where they cannot be avoided special provision should be made for removing the air which collects.

104. Special Problems.—Pipe lines may be composed of pipes of several diameters connected in series, or they may branch in different ways so as to divide the flow, thus presenting a

variety of problems. Oftentimes such problems may be solved more readily by trial solutions, though some formulas may be derived which are of assistance. A few special cases are given in the following pages. Problems of this type are encountered frequently in designing mains for city water supplies.

If the pipes are long (1000 diameters or more) the minor losses will ordinarily be comparatively small and are usually neglected. If, however, it is desired to include these losses, a solution should be made first neglecting them and then correcting the results to include them.

105. Branching Pipe Connecting Reservoirs at Different Elevations.—A, B and C are three reservoirs connected by pipes 1, 2 and 3, as shown in Fig. 101. Let l_1, d_1, Q_1 and v_1

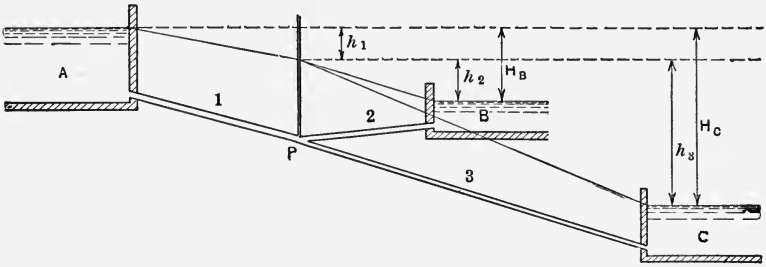

FIG. 101.—Branching pipe connecting three reservoirs.

represent, respectively, the length, diameter, discharge and mean velocity for pipe 1, and the same symbols with subscripts 2 and 3, the corresponding terms for pipes 2 and 3. If a piezometer is assumed to be at the junction P, the water surface in the tube will be a certain distance, h_1, below the surface in reservoir A. The surface of reservoir B is a distance $H_B=h_1+h_2$ below that of reservoir A and the surface of reservoir C is $H_C=h_1+h_3$ below the surface of reservoir A. If $h_1<H_B$, reservoir A will supply reservoirs B and C. If $h_1>H_B$, reservoirs A and B will supply reservoir C. There are many problems suggested by this figure, in which certain quantities are given with others to be determined. Methods of solving three of these problems are given.

Case 1.—Having given the lengths and diameters of all pipes, and elevations of the three reservoirs; to determine Q_1, Q_2 and Q_3.

This problem is most conveniently solved by trial. Assume Q_1 and solve for h_1. Then using H_B minus this computed value of h_1 for h_2, the loss of head due to friction in pipe 2, solve for Q_2. Similarly, using H_C minus the computed value of h_1 for friction loss, h_3, in pipe 3, solve for Q_3. Evidently $Q_1=Q_2+Q_3$, Q_2 being negative if the direction of flow is from B toward P. The correct value of Q_1 will lie between the assumed value and the computed value of Q_2+Q_3. Continue to assume new values of Q_1, between these limits, and repeat computations until $Q_1=Q_2+Q_3$.

It may be found helpful in making assumptions to plot computed values of Q_1, Fig. 102, against the error made in each assumption, that is, against $Q_1-(Q_2+Q_3)$. The resulting difference may be either plus or minus. If the assumed values of Q_1 are well selected they will define a curve whose intersection with the Q_1-axis will give the discharge as accurately as is usually required. The points should be on both sides of the Q_1-axis and preferably one of the points should be quite close to it. Usually not more than three trial solutions will be necessary.

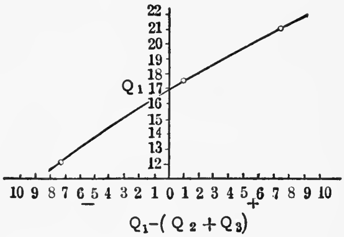

FIG. 102.

This problem may also be solved analytically. Assuming any formula for pipe friction, as, for example, the Chezy formula,

$$h_1=f_1\frac{l_1}{d_1}\frac{v_1^2}{2g};$$

also

$$h_2=f_2\frac{l_2}{d_2}\frac{v_2^2}{2g};$$

and

$$h_3=f_3\frac{l_3}{d_3}\frac{v_3^2}{2g}.$$

From Fig. (101),

$$H_B=h_1+h_2,$$

and substituting the above values of h_1 and h_2,

$$H_B=f_1\frac{l_1}{d_1}\frac{v_1^2}{2g}+f_2\frac{l_2}{d_2}\frac{v_2^2}{2g}, \quad . \quad . \quad . \quad . \quad . \quad (37)$$

and in a similar manner,

$$H_C=f_1\frac{l_1}{d_1}\frac{v_1^2}{2g}+f_3\frac{l_3}{d_3}\frac{v_3^2}{2g}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

also since $Q_1=Q_2+Q_3$

$$d_1^2v_1=d_2^2v_2+d_3^2v_3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

By solving equations (37), (38) and (39) simultaneously, v_1, v_2 and v_3 may be determined if the other quantities are given. Since f_1, f_2 and f_3 are dependent upon v_1, v_2 and v_3, respectively for their values, the equations must first be solved with assumed friction coefficients, to be corrected after the first solution for velocities has been completed. With these corrected values of f_1, f_2 and f_3, another solution of the equations for more accurate values of v_1, v_2 and v_3 may be made.

Case 2.—Having given the lengths and diameters of all pipes, Q_1, and the elevations of water surfaces in reservoir A and one of the other reservoirs as B; to determine the elevation of water surface in reservoir C.

Using Q_1, determine the lost head, h_1, in pipe 1. Then $h_2=H_B-h_1$ is the lost head in pipe 2, using which, Q_2 may be computed. Q_2 will be plus or minus depending upon whether the direction of flow in pipe 2 is towards B or P. Then $Q_3=Q_1-Q_2$. With Q_3 determined, the head lost in pipe 3 may be computed, and the elevation of water surface in reservoir C obtained.

Case 3.—Having given the lengths of all pipes, the elevations of water surfaces in all reservoirs, Q_1, and the diameters of two pipes as d_1 and d_2; to determine d_3.

Determine h_1, Q_2 and Q_3 as for Case 2. Then with Q_3 and $h_3=H_C-h_1$ known, compute d_3.

106. Compound Pipe Connecting Two Reservoirs.—The reservoirs A and B are connected by a system of pipes as shown in Fig. 103. Pipe 1 leading from reservoir A divides at S into pipes 2 and 3 which join again at T. Pipe 4 leads from the junction T to a reservoir B. Let l_1, d_1 and v_1 be respectively the length, diameter and mean velocity for pipe 1, and the same symbols with subscripts 2, 3 and 4 the corresponding quanitites for pipes 2, 3 and 4. Q_2 and Q_3 are the respective discharges for pipes 2 and 3, the sum of which discharges equals Q, the total discharge through pipes 1 and 4. Assuming piezometer tubes

at S and T, H is the total head lost in the system of pipes, h_1 is the head lost in pipe 1, $h_2=h_3$ is the head lost in pipes 2 or 3, and h_4 is the head lost in pipe 4. Before taking up any of the special problems suggested by this figure a general analysis will be given.

Any of the formulas for determining loss of head due to friction may be employed. It will be found advantageous to use a formula whose coefficient does not vary with the diameter,

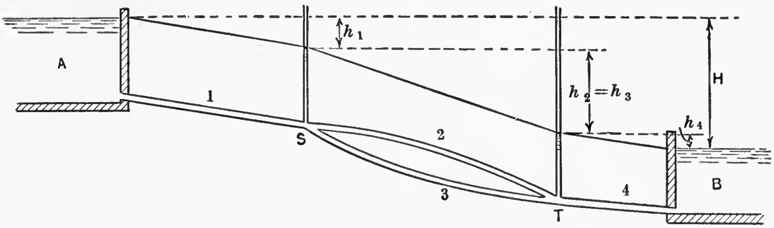

FIG. 103.—Compound pipe.

but a fractional exponent for v will be objectionable. Formula (20) has, therefore, been selected. Since $h_2=h_3$, from formula (20),

$$K_2\frac{l_2}{d_2^{1.25}}\frac{v_2^2}{2g}=K_3\frac{l_3}{d_3^{1.25}}\frac{v_3^2}{2g}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (40)$$

also

$$v_2=\frac{Q_2}{a_2}=\frac{4Q_2}{\pi d_2^2}= \quad \text{and} \quad v_3=\frac{Q_3}{a_3}=\frac{4Q_3}{\pi d_3^2},$$

writing these values of v_2 and v_3 in equation (40),

$$K_2\frac{l_2}{d_2^{1.25}}\frac{16Q_2^2}{\pi^2 d_2^4}\frac{1}{2g}=K_3\frac{l_3}{d_3^{1.25}}\frac{16Q_3^2}{\pi^2 d_3^4}\frac{1}{2g}. \quad . \quad . \quad . \qquad (41)$$

Since in formula (20) K varies only with the velocity and not with the diameter, the error introduced by assuming $K_2=K_3$ will not be important unless v_1 and v_2 are widely different. Assuming them equal and canceling,

$$\frac{l_2}{d_2^{5.25}}Q_2^2=\frac{l_3}{d_3^{5.25}}Q_3^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (42)$$

or

$$Q_3=Q_2\sqrt{\frac{l_2}{l_3}\left(\frac{d_3}{d_2}\right)^{5.25}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (43)$$

Placing

$$F=\sqrt{\frac{l_2}{l_3}\left(\frac{d_3}{d_2}\right)^{5.25}}, \quad \ldots \ldots \quad (44)$$

$$Q_3=FQ_2, \quad \ldots \ldots \quad (45)$$

and since

$$Q_2+Q_3=Q, \quad \ldots \ldots \quad (46)$$

$$Q_2+FQ_2=Q, \quad \ldots \ldots \quad (47)$$

and

$$Q_2=\frac{Q}{1+F}. \quad \ldots \ldots \quad (48)$$

Therefore to determine approximately (within the limits of the error introduced in assuming K to be a constant) the quantity of water passing through pipe 2 divide the total discharge by $1+F$ and the total discharge minus Q_2 gives Q_3.

The expression for loss of head may be written,

$$H=h_1+h_2+h_4. \quad \ldots \ldots \quad (49)$$

The losses of head in the pipes 1, 2 and 4 may be expressed by either formula (16) or (20). Using (20), the expression for lost head becomes

$$H=K_1\frac{l_1}{d_1^{1.25}}\frac{v_1^2}{2g}+K_2\frac{l_2}{d_2^{1.25}}\frac{v_2^2}{2g}+K_4\frac{l_4}{d_4^{1.25}}\frac{v_4^2}{2g}. \quad . \quad (50)$$

But

$$v_1=\frac{Q}{a_1}=\frac{4Q}{\pi d_1^2}, \quad \ldots \ldots \quad (51)$$

$$v_2=\frac{Q_2}{a_2}=\frac{4Q_2}{\pi d_2^2}=\frac{4Q}{(1+F)\pi d_2^2}, \quad \ldots \ldots \quad (52)$$

and

$$v_4=\frac{Q}{a_4}=\frac{4Q}{\pi d_4^2}. \quad \ldots \ldots \quad (53)$$

Substituting these values of v_1, v_2 and v_4 in equation (50) and reducing

$$H=\frac{16Q^2}{2g\pi^2}\left(K_1\frac{l_1}{d_1^{5.25}}+K_2\frac{l_2}{(1+F)d_2^{5.25}}+K_4\frac{l_4}{d_4^{5.25}}\right). \quad (54)$$

If formula (16) in place of (20) had been used in writing equation (50) the above formula would be

$$H=\frac{16Q^2}{2g\pi^2}\left(f_1\frac{l_1}{d_1^5}+f_2\frac{l_2}{(1+F)d_2^5}+f_4\frac{l_4}{d_4^5}\right). \quad \ldots \quad (55)$$

In formulas (54) and (55) the coefficients are not constants. K_1, K_2, and K_4 vary with the velocity and f_1, f_2, and f_4 vary both with the velocity and diameter. Formula (54) will be more convenient in solving problems where all diameters are not known since K_1, K_2, and K_4 do not depend upon the diameter for their value. Three types of problems are explained below.

Case 1.—Having given the lengths and diameters of all pipes and the total lost head; to determine Q.

Determine F by formula (44), then determine an approximate value of Q from formula (54) or (55) estimating the velocities in pipes 1, 2 and 4 for obtaining trial values of K_1, K_2 and K_4, or f_1, f_2 and f_4. If the estimated velocities were not too much in error this solution may give the value of Q as accurate as is desired; otherwise with the value of Q obtained by the first solution determine the velocities and corresponding values of coefficients in the three pipes and again solve equation (54) or (55) for Q. The second solution should always give Q within the desired degree of accuracy.

To check the result, from the computed value of Q determine h_1 and h_4, then using $H-(h_1+h_4)$ as the friction loss in pipes 2 and 3 compute Q_2 and Q_3. The results obtained should show Q approximately equal to Q_2+Q_3. If closer agreement between the computed values of Q and Q_2+Q_3 is required than may be obtained readily by this method the results may be adjusted by trial solutions. Since there is always uncertainty as to the proper value of coefficients to use, it is not usually desirable to work for too close an agreement.

If preferred this problem may be solved entirely by trial but it will save time in trial solutions to determine first by formulas (44) and (48) the portion of the total flow that passes through one of the branching pipes. Then successive values of Q may be assumed and the lost head in each pipe computed until the sum of the losses in the three pipes equals the total lost head. A final check should be made to see that Q equals approximately Q_2+Q_3.

Case 2.—Having given the discharge, diameters and lengths of all pipes; to determine the total lost head.

The lost head, H, may be determined from equations (44) and (54).

If preferred Q_2 may be first obtained from formulas (44) and

(48) and Q_3 from formula (45). Then using these discharges, compute the head lost in pipes 2 and 3. These should be equal. If the computations do not show them equal adjust the discharges by trial (reducing the discharge in the pipe showing the greater loss of head and increasing by the same amount the discharge in the other pipe), and again compute the head lost in each pipe. Repeat the assumption and computations until the losses of head in each of the pipes become the same or until the agreement is close enough for the purpose. This loss of head plus the loss of head in pipes 1 and 4, which may be computed in the usual manner, gives, H, the total lost head.

Case 3.—Having given the discharge, the total lost head, the lengths of all pipes and the diameters of three pipes; to determine the other diameter.

Assume that the diameter of pipe 2 is to be determined. Compute the head lost in pipes 1 and 4 by one of the formulas for determining loss of head due to friction. Deduct from the total lost head the sum of these computed losses. With this difference, which is the head lost in each of pipes 2 and 3, determine Q_3. Then, $Q_2=Q-Q_3$. With Q_2 known, and the lost head determined as described above, compute the diameter of pipe 2.

If the diameter of one of the single pipes, as for example, pipe 4, is to be determined, compute the head lost in pipe 1, as described in the preceding paragraph and also the head lost in the branching pipes 2 and 3 as described under Case 2. The difference between the total lost head and the sum of the above losses is the head lost in pipe 4, from which the diameter of this pipe may be computed.

107. Pipes of More than One Diameter Connected in Series.—Fig. 104 represents a pipe of three diameters with lengths l_1, l_2 and l_3; diameters d_1, d_2 and d_3; and velocities v_1, v_2 and v_3. The total loss of head, H, assuming formula (19) for friction loss is

$$H=K_1\frac{l_1}{d_1^{1.25}}\frac{v_1^2}{2g}+K_2\frac{l_2}{d_2^{1.25}}\frac{v_2^2}{2g}+K_3\frac{l_3}{d_3^{1.25}}\frac{v_3^2}{2g}; \quad . \quad . \quad . \quad (56)$$

also

$$Q=av=\frac{\pi d_1^2}{4}v_1=\frac{\pi d_2^2}{4}v_2=\frac{\pi d_3^2}{4}v_3, \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

substituting the values of v_1, v_2 and v_3 obtained from (57) in equation (56) and transposing the following formula is obtained:

$$H=\frac{16Q^2}{2g\pi^2}\left(K_1\frac{l}{d_1^{5.25}}+K_2\frac{l_2}{d_2^{5.25}}+K_3\frac{l_3}{d_3^{5.25}}\right). \quad . \quad . \quad (58)$$

From this formula with all quantities but one given, the unknown quantity may be obtained. If the velocity is not known, values

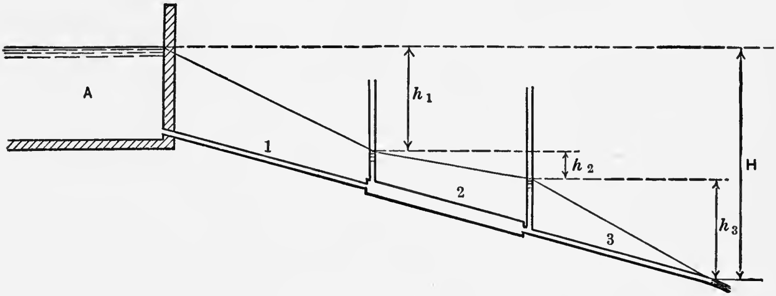

FIG. 104.—Pipe of three diameters.

of K_1, K_2 and K_3 must be assumed and corrected from a trial solution. A second solution of the problem may then be made.

PROBLEMS

1. A new cast-iron pipe 1200 ft. long and 6 in. in diameter carries 1.3 cu. ft. per second. Determine the frictional loss.

2. Determine the discharge of the pipe described in Problem 1 if it discharges under a head of 80 ft.

3. What diameter of new cast-iron pipe, 1 mile long is required to discharge 4.5 cu. ft. per second under a head of 50 ft.?

4. What diameter of concrete pipe 8000 ft. long is required to discharge 40 cu. ft. per second under a head of 8 ft.?

5. Determine the loss of head due to sudden enlargement if a pipe carrying 2.0 cu. ft. per second suddenly changes from a diameter of (*a*) 6 in. to 8 in., (*b*) 6 in. to 12 in., and (*c*) 6 in. to 18 in. Also determine the difference in pressure resulting from these changes.

6. Solve Problem 5 if the direction of flow is reversed in each case.

7. A cast-iron pipe 12 in. in diameter and 100 ft. long having a sharp-cornered entrance draws water from a reservoir and discharges into the air. What is the difference in elevation between the water surface in the reservoir and the discharge end of the pipe if the rate of discharge is 16.0 cu. ft. per second?

8. If the pipe described in Problem 7 connects two reservoirs, both ends being sharp-cornered and submerged, other conditions remaining the same,

determine the difference in elevation of the water surfaces in the two reservoirs.

9. A wood-stave pipe having a diameter of 48 in. is laid on a down-grade of 6 ft. per mile. Determine the difference in pressure between two points 1 mile apart if the discharge is 45 cu. ft. per second.

10. A wood-stave pipe, 500 ft. long, is to be designed to carry 60 cu. ft. per second across a ravine so as to connect the ends of an open flume. If the difference in elevation between the water surfaces in the two ends of the flume is to be 5 ft., determine the necessary diameter of pipe, assuming abrupt changes in section, and neglecting the effect of velocity in the flume.

11. A concrete-pipe culvert 90 ft. long and 3 ft. in diameter is built through a road embankment. The culvert is laid on a grade of 1 ft. per 100 ft. Water is backed up to a depth of 5 ft. above the top of the pipe at the entrance and at the outlet the top of the pipe is submerged to a depth of 2 ft. Assume a sharp-cornered entrance. What is the discharge?

12. In Problem 11 (assuming all other conditions to be the same), what diameter of pipe will be required to discharge 100 cu. ft. per second?

13. In Problem 11 (assuming all other conditions to be the same) what will be the depth of water above the top of the pipe at its entrance when the culvert is discharging 50 cu. ft. per second?

14. A pipe line is to be laid connecting two tangents which intersect at an angle of 90°. Between two points on these tangents, each distant 100 ft. from their point of intersection, will the total loss in head be less if a bend having a radius of 6 ft. or one having a radius of 50 ft. is used?

15. Three new cast-iron pipes are connected in series as shown in Fig. 104. The first has a diameter of 12 in. and a length of 1200 ft.; the second has a diameter of 24 in. and a length of 2000 ft.; and the third has a diameter of 18 in. and a length of 1500 ft. If the discharge is 8 cu. ft. per second, determine the lost head neglecting the minor losses.

16. If, in Problem 15, the total lost head in the three pipes is 45 ft., neglecting the minor losses, determine the discharge.

17. If the three pipes, described in Problem 15, have lengths of 500 ft. each, the entrance and all changes in section being sharp-cornered, determine the total lost head when the discharge is 8 cu. ft. per second.

18. Referring to Fig. 101, page 166, if pipes 1, 2 and 3 have diameters of 24 in., 12 in. and 18 in., and lengths of 1200 ft., 500 ft. and 1000 ft., respectively, determine the discharge through pipe 1, if $H_B = 12$ ft. and $H_C = 20$ ft. Neglect minor losses.

19. In Problem 18, determine H_C, if the discharge through pipe 1 is 20 cu. ft. per second, all other conditions remaining the same.

20. In Problem 18, determine the diameter of pipe 3 if the discharge through pipe 1 is 24 cu. ft. per second, all other conditions remaining constant.

21. Referring to Fig. 103, page 169, if pipes 1, 2, 3 and 4 have diameters of 36 in., 18 in., 24 in. and 30 in., and lengths of 3000 ft., 2000 ft., 2400 ft. and 1500 ft., respectively, determine H, if the discharge through the system is 60 cu. ft. per second. Neglect the minor losses.

22. In Problem 21, determine the discharge if H is 12 ft., other conditions remaining the same.

23. In Problem 21, determine the necessary diameter of pipe 3 if H is 15 ft., other conditions remaining the same.

24. Three smooth rubber-lined fire hose, each 200 ft. long and $2\frac{1}{2}$ in. in diameter and having 1 in. nozzles, are connected to a 6-in. fire hydrant If, for the nozzles $C_c=1$ and $C_v=0.97$, determine the necessary pressure in the hydrant in order to throw streams 100 ft. high, the nozzles being 10 ft. above the hydrant.

25. Determine the height of streams that can be thrown if the pressure in the hydrant is 70 lbs. per square inch, all other conditions remaining as stated in Problem 24.

CHAPTER X

FLOW OF WATER IN OPEN CHANNELS

108. Description and Definition.—An *open channel* is a conduit which conveys water without exerting any pressure, above atmospheric pressure, other than that due to the actual weight of water carried. Water therefore ordinarily flows in an open channel with a free water surface, though for an enclosed conduit, like a sewer flowing full, water may touch the top surface without exerting pressure. In this case it is classed as an open channel. Examples of open channels are rivers, canals, flumes, and sewers and aqueducts when carrying water not under pressure.

Open channels have various forms of cross-section. Artificial channels are commonly of rectangular, trapezoidal, or circular cross-section, while natural streams have irregular channels. Though friction losses in open channels follow the same general laws as in pipes and pipe formulas could be adapted to them, special friction formulas for open channels are usually employed.

109. Wetted Perimeter and Hydraulic Radius.—The *wetted perimeter* of any conduit is the line of intersection of its wetted surface with a cross-sectional plane. In Fig. 105 the wetted perimeter is the length of the broken line *abcd*. In a circular conduit flowing part full, as a sewer, the wetted perimeter is the arc of a circular segment and in a natural stream, Fig. 116, it is the irregular line *abcde*.

Fig. 105.—Cross-section of trapezoidal channel.

The *hydraulic radius* of any section of a channel is its area divided by the wetted perimeter. All open channel formulas express the velocity as a function of the hydraulic radius.

110. Friction and Distribution of Velocities.—As described under pipes (Art. 91) there is friction between the moving water and the surfaces of any conduit. If there were no other influences

the maximum velocity would ordinarily occur at places most remote from the surfaces of the conduit which produce friction, and consequently at the water surface. Owing to surface tension, however, there is a resistance to flow at the surface of the water, and the maximum velocity occurs at some distance below the surface. Under ideal conditions, where there are no disturbing influences of any kind, the distribution of velocities in a regular channel will be uniform and similar on either side of the center. There are always, however, some irregularities in every channel sufficient to prevent a uniform distribution of velocities. The lines of equal velocity plotted from a large number of velocity measurements for the Sudbury conduit near Boston, Fig. 106, shows a more regular distribution of velocities than will be found in most channels. The distribution of velocities in a river of irregular cross-section, as

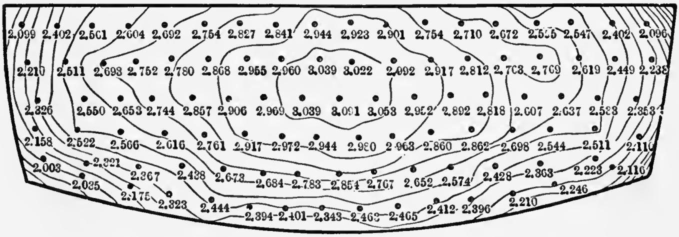

FIG. 106.—Distribution of velocities in Sudbury conduit.

determined from velocity measurements with a current meter, is shown in the upper portion of Fig. 107. The figures show the velocities obtained at the points, where measurements were made and the irregular lines are interpolated equal velocity lines.

The curves in the lower portion of Fig. 107 show the distribution of velocities in vertical lines. These curves are called *vertical velocity curves* and the velocities from which they are plotted are called *velocities in the vertical.* The following properties of vertical velocity curves have been determined from the measurement of velocities of a large number of streams and a study of the curves plotted from them.

(*a*) Vertical velocity curves have approximately the form of parabolas with horizontal axes passing through the thread of maximum velocity. In general the maximum velocity occurs somewhere between the water surface and one-third of the depth,

the distance from the surface to the point of maximum velocity being at a greater proportional depth for greater depths of water. For shallow streams the maximum velocity is very near to the surface while for very deep streams it may lie at about one-third

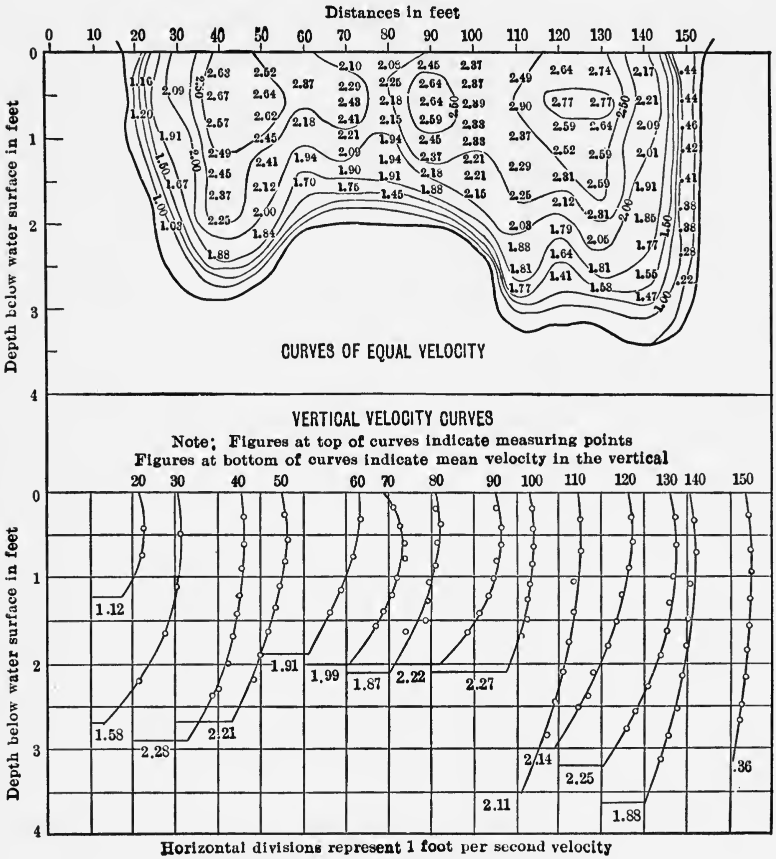

FIG. 107.—Velocities in natural stream.

of the depth. A strong wind blowing either upstream or downstream will affect the distribution of velocities in the vertical.

(*b*) The mean velocity in the vertical is ordinarily found at a distance below the surface varying from 0.55 to 0.65 of the depth. The velocity at 0.6 depth is usually within 5 per cent of the mean velocity.

(*c*) The mean of the velocities at 0.2 and 0.8 depth usually gives the mean velocity in the vertical within 2 per cent.

(*d*) The mean velocity in the vertical is ordinarily from 0.80 to 0.95 of the surface velocity. The smaller percentage applies to the shallower streams.

The above properties of vertical velocity curves are made use of in measuring the discharges of streams. Mean velocities in successive verticals are first obtained by measuring the velocity at 0.6 of the depth in each vertical or, where greater accuracy is required, by taking the mean of the velocities at 0.2 and 0.8 of the depth. The mean of the velocities in any two adjacent verticals is considered as the mean velocity of the water between these verticals. The area between the verticals having been determined, the discharge through this portion of the cross-section of the stream is the product of this area and the mean velocity. The sum of all discharges between successive verticals is the total discharge.

The mean velocity in the vertical may be obtained by taking the mean of several velocity measurements. This method is more laborious, however, and does not give the mean velocity appreciably more accurately than that obtained by taking the mean of velocities at 0.2 and 0.8 of the depth.

The distribution of velocities in an ice-covered stream, Fig. 108, is modified by the effect of friction between the water and the ice. The amount of this friction exceeds the skin friction of a free water surface and the maximum velocity therefore occurs nearer mid-depth. The mean velocity in the vertical for an ice-covered stream is not at 0.6 depth but the mean of velocities at 0.2 and 0.8 depth gives approximately the mean velocity the same as for a stream with a free-water surface.

111. Energy Contained in Water in an Open Channel.—This subject has been discussed (Art. 72) in connection with the velocity of approach for weirs, and the energy of water in a pipe is discussed in Art. 92. In open channels where velocities in different parts of a cross-section are not the same, the total kinetic energy contained in the water flowing past any cross-section is

$$KE = \alpha W \frac{v^2}{2g}, \quad \ldots \ldots \ldots \quad (1)$$

W being the total weight of water passing the cross-section in one second, v the mean velocity, and α an empirical coefficient depend-

ing for its value upon the distribution of velocities in the channel but always greater than unity. The range of variation in α has not been determined but in channels with unobstructed flow it probably lies between 1.1 and 1.2. This matter is not of great importance in ordinary hydraulic problems. Velocity head in open channels is commonly taken as the head due to the mean

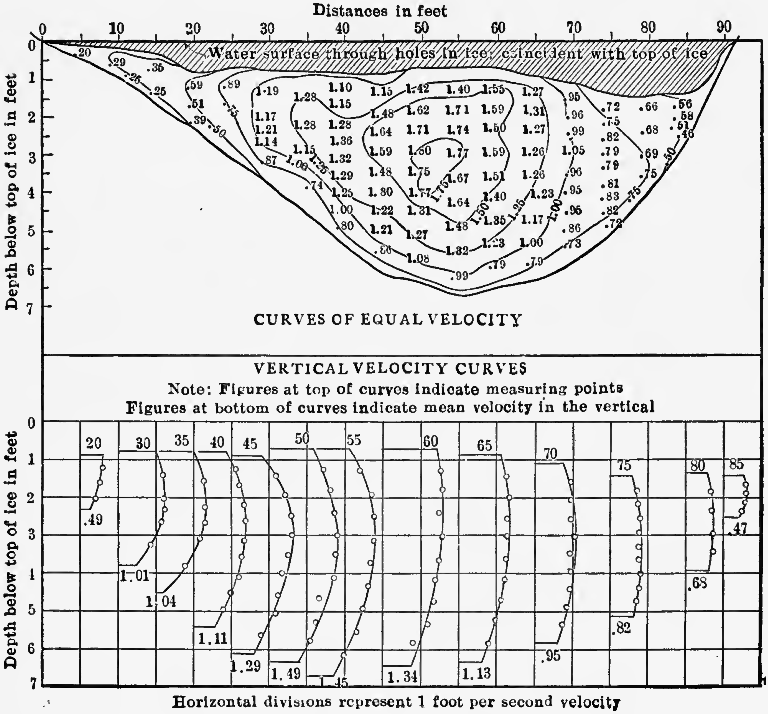

Fig. 108.—Velocities in ice-covered stream.

velocity, that is, α is assumed to equal unity. The slight error introduced by this assumption may be partially eliminated by a proper selection of coefficients.

112. Continuity of Flow in Open Channels.—In any open channel fed by a constant supply of water there is the same rate of flow past every cross-section. In pipes flowing full, Art. 93, since water is practically incompressible, it is not necessary that the supply be constant in order to have the same rate of flow past

every cross-section at the same instant; but in open channels the water is not confined and a variation in the rate of supply will cause unequal rates of flow past different sections of the channel. In other words, in order to have continuity of flow in an open channel it is necessary at the same time to have steady flow, but in pipes flowing full there is always continuity of flow regardless of whether the flow is steady or variable.

When continuity of flow exists in an open channel, the mean velocities are equal at all cross-sections having equal areas but if the areas are not equal the velocities are inversely proportional to the areas of the respective cross-sections.

Thus, if a_1 and v_1, and a_2 and v_2 are respectively areas and mean

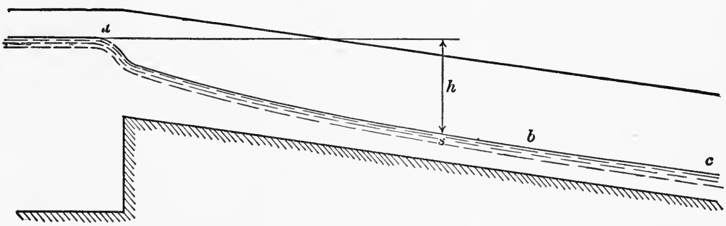

FIG. 109.—Open channel with accelerating velocity.

velocities at any two cross-sections in an open channel where continuity of flow exists,

$$a_1 v_1 = a_2 v_2$$

and

$$\frac{v_1}{v_2} = \frac{a_2}{a_1}.$$

113. Loss of Head.—Figs. 109 and 110 represent channels of constant cross-sections receiving water at uniform rates from reservoirs. Potential energy changes to kinetic energy as movement of the water takes place. As the water flows down these channels a portion of the energy is lost by friction between the water and the surfaces of the channels. If there were no friction losses of any kind the velocity, v_t at any section, s, would be $v_t = \sqrt{2gh}$, in which h is the vertical distance between the water surface in the reservoir and the water surface at the section of the channel. The velocity of the water would thus continue to accelerate as long as the downward slope of the channel continued.

The actual conditions of flow as modified by friction for the case illustrated in Fig. 109 are as follows. The water receives a certain initial velocity at *a*, (Art. 85) where the channel leaves the reservoir. This channel, as indicated in the figure, has a slope steeper than is required to carry the water with the initial velocity which it receives at *a*. The velocity therefore accelerates for some distance, but a portion of the energy which the water contains is used in overcoming friction. As the water proceeds down the channel at a continually increasing velocity, a point is finally reached (since frictional resistance increases with the velocity, as has already been shown for pipes and will be shown for open channels) beyond which the energy used in overcoming friction in any reach exactly equals the potential energy contained in the water within the reach by reason of the slope of the channel. After this point has been reached, approximately at *b* in the figure,

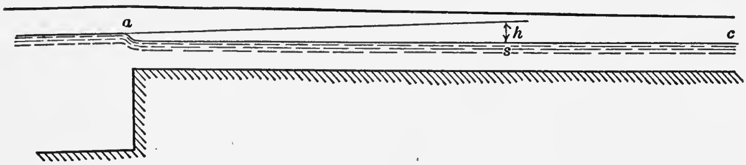

Fig. 110.—Open channel with constant velocity.

the water will flow to *c* and beyond with a constant velocity as long as channel conditions remain unchanged. As water moves from *a* to *b* a portion of its potential energy is continually being changed to kinetic energy and the remainder is used in overcoming friction. Between *b* and *c* the kinetic energy remains constant and all of the potential energy is used in overcoming friction.

Fig. 110 illustrates the case where the slope is no greater than is required to carry the water at the initial velocity which it receives at *a*. Under these circumstances, as long as the channel conditions remain constant, the velocity in the channel will be the same at all sections. In other words the potential energy of the water will all be used in overcoming friction and the kinetic energy will remain constant. This case is the one most commonly encountered in open channel problems.

In this connection it is important to bear in mind that the velocity remains constant only so long as the channel conditions

remain constant. Any change in the size of the channel changes the velocity. An increase or decrease in the slope of the channel causes a corresponding increase or decrease in velocity. The velocity is also modified by any conditions affecting frictional resistance. It will be noted that velocity conditions for open channels are different than for pipes. In a pipe flowing full, under a constant head, the water being confined and considered incompressible, the mean velocity can change only when the diameter of the pipe changes. In an open channel, however, the water being unconfined, the velocity changes with every change in channel conditions.

Losses of head in open channels are in every respect analogous to losses of head in pipes. In addition to the loss of head due to friction between the moving water and the surface of the channel there is a loss of head wherever the velocity of water or direction of flow is changed. The same symbols will be used to represent losses of head in open channels that are used to designate the corresponding losses in pipes (Art. 94). These losses for open channels are as follows:

(*a*) A continuous loss of head throughout the channel due to friction between the moving water and the surface of the channel and to viscosity. This loss is commonly referred to as the *loss of head due to friction.*

(*b*) A *loss of head at entrance* to the channel, that is, where a channel takes water from a reservoir or other body of comparatively still water.

(*c*) A *loss of head at discharge,* that is, where a channel discharges into a reservoir or other body of comparatively still water.

(*d*) A *loss of head due to contraction* where a channel changes to a smaller cross-sectional area causing an increase in velocity. The loss of head at entrance to a channel (referred to under (*b*) above) is a special case of this loss.

(*e*) A *loss of head due to enlargement* where a channel changes to a larger cross-sectional area causing a decrease in velocity. The loss of head at discharge (referred to under (*c*) above) is a special case of this loss.

(*f*) A *loss of head due to obstructions* of any kind in a channel, such as gates, bridge piers or submerged weirs.

(*g*) A *loss of head due to curves* in a channel in addition to the loss which occurs in an equal length of straight channel.

114. Hydraulic Gradient or Water Surface.—The hydraulic gradient (Art. 95) of an open channel coincides with the water surface. In the case of steady, uniform flow, the water surface is parallel to the bottom of the channel. Any changes in channel conditions which will cause either an increase or decrease in velocity will cause a change in the elevation of the water surface the same as a change in velocity in a pipe will cause a change in the hydraulic gradient. Changes in the hydraulic gradient of a pipe line resulting from changes in section are frequently of minor importance and need be considered only insofar as they affect the total loss of head, while for an open channel the effect of any changes in the cross-section should be thoroughly understood and designs for the transition of the water from one velocity to another, should be worked out with great care.

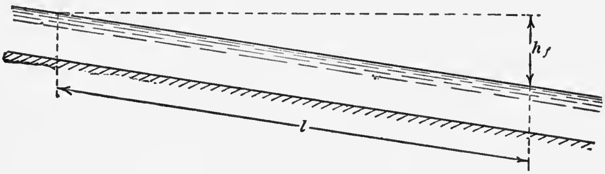

FIG. 111.

Failure to provide properly for changes in elevation of water surface at the place where such changes occur may make it necessary for the channel to carry water at a depth other than that for which it was designed and thus interfere with its satisfactory operation.

115. Loss of Head Due to Friction in Open Channels.—Fig. 111 represents the condition of steady, uniform flow in a straight channel. Since all of the head, h_f, is used in overcoming friction in the distance, l, this lost head is a measure of the resistance to flow. The ratio h_f/l is called the slope, and is represented by the symbol s. Since friction losses in open channels and pipes are of the same character they are governed by the same laws. To make the general laws as stated for pipes on page 145 apply to open channels it is necessary only to substitute, respectively, the words channel and hydraulic radius for pipe and diameter. It is evident, therefore, that the base formulas for pipes apply equally to open channels. Formula (14), page 146, is in the form generally used

for open channels. For convenience of reference it is here repeated.

$$v = K''' r^y s^z. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The further consideration of friction losses in open channels must be purely empirical. Numerical values of empirical coefficients and exponents and, if necessary, modifications in the form of the base formula must be derived from experimental data. A few of the more commonly used open channel formulas are given in the following pages.

116. The Chezy Formula.—This formula as stated in the preceding chapter (Art. 97) is, with proper modification, applicable either to open channels or to pipes, though it was originally designed for open channels. The formula as written by Chezy is

$$v = C\sqrt{rs}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which v is the mean velocity, r is the hydraulic radius, and s is the slope of water surface. The Chezy formula is of the same form as formula (2); y and z each being equal to $\frac{1}{2}$ and C being written for K'''. The value of the coefficient C varies with the characteristics of the channel. In the form given it is not therefore readily adaptable to open channels but with modifications it forms the basis of most of the formulas in common use.

117. The Kutter Formula.—An elaborate investigation of all available records of measurements of flow in open channels was made by Ganguillet and Kutter,[1] Swiss engineers, in 1869. As a result of their study they deduced the following empirical formula, commonly called the Kutter formula, for determining the value of C in the Chezy formula.

$$C = \frac{41.65 + \dfrac{0.00281}{s} + \dfrac{1.811}{n}}{1 + \dfrac{n}{\sqrt{r}}\left(41.65 + \dfrac{0.00281}{s}\right)}. \quad . \quad . \quad . \quad . \quad (4)$$

In the above formula, C is expressed as a function of the hydraulic radius, r, and slope, s, as well as the coefficient of roughness, n, whose value increases with the degree of roughness of the channel.

[1] GANGUILLET and KUTTER: Flow of Water in Rivers and Other Channels. Translation by HERRING and TRAUTWINE, John Wiley & Sons, Publishers.

Values of C from Kutter's Formula

Slope	n	Hydraulic Radius r in Feet														
		0.2	0.3	0.4	0.6	0.8	1.0	1.5	2.0	2.5	3.0	4.0	6.0	8.0	10.0	15.0
.00005	0.010	87	98	109	123	133	140	154	164	172	177	187	199	207	213	220
	.012	68	78	88	98	107	113	126	135	142	148	157	168	176	182	189
	.015	52	58	66	76	83	89	99	107	113	118	126	138	145	150	159
	.017	43	50	57	65	72	77	86	93	98	103	112	122	129	134	142
	.020	35	41	45	53	59	64	72	80	84	88	95	105	111	116	125
	.025	26	30	35	41	45	49	57	62	66	70	78	85	92	96	104
	.030	22	25	28	33	37	40	47	51	55	58	65	74	78	83	90
.0001	0.010	98	108	118	131	140	147	158	167	173	178	186	196	202	206	212
	.012	76	86	95	105	113	119	130	138	144	148	155	165	170	174	180
	.015	57	64	72	81	88	93	103	109	114	118	125	134	140	143	150
	.017	48	55	62	70	75	80	88	95	99	104	111	118	125	128	135
	.020	38	45	50	57	63	67	75	81	85	88	95	102	107	111	118
	.025	28	34	38	43	48	51	59	64	67	70	77	84	89	93	98
	.030	23	27	30	35	39	42	48	52	55	59	64	72	75	80	85
.0002	0.010	105	115	125	137	145	150	162	169	174	178	185	193	198	202	206
	.012	83	92	100	110	117	123	133	139	144	148	154	162	167	170	175
	.015	61	69	76	84	91	96	105	110	114	118	124	132	137	140	145
	.017	52	59	65	73	78	83	90	97	100	104	110	117	122	125	130
	.020	42	48	53	60	65	68	76	82	85	88	94	100	105	108	113
	.025	30	35	40	45	50	54	60	65	68	70	76	83	86	90	95
	.030	25	28	32	37	40	43	49	53	56	59	63	69	74	77	82
.0004	0.010	110	121	128	140	148	153	164	171	174	178	184	192	197	198	203
	.012	87	95	103	113	120	125	134	141	145	149	153	161	165	168	172
	.015	64	73	78	87	93	98	106	112	115	118	123	130	134	137	142
	.017	54	62	68	75	80	84	92	98	101	104	110	116	120	123	128
	.020	43	50	55	61	67	70	77	83	86	88	94	99	104	106	110
	.025	32	37	42	47	51	55	60	65	68	70	75	82	85	88	92
	.030	26	30	33	38	41	44	50	54	57	59	63	68	73	75	80
.001	0.010	113	124	132	143	150	155	165	172	175	178	184	190	195	197	201
	.012	88	97	105	115	121	127	135	142	145	149	154	160	164	167	171
	.015	66	75	80	88	94	98	107	112	116	119	123	130	133	135	141
	.017	55	63	68	76	81	85	92	98	102	105	110	115	119	122	127
	.020	45	51	56	62	68	71	78	84	87	89	93	98	103	105	109
	.025	33	38	43	48	52	55	61	65	68	70	75	81	84	87	91
	.030	27	30	34	38	42	45	50	54	57	59	63	68	72	74	78
.01	0.010	114	125	133	143	151	156	165	172	175	178	184	190	194	196	200
	.012	89	99	106	116	122	128	136	142	145	149	154	159	163	166	170
	.015	67	76	81	89	95	99	107	113	116	119	123	129	133	135	140
	.017	56	64	69	77	82	86	93	99	103	105	109	115	118	121	126
	.020	46	52	57	63	68	72	78	84	87	89	93	98	102	105	108
	.025	34	39	44	49	52	56	62	65	68	70	75	80	83	86	90
	.030	27	31	35	39	43	45	51	55	58	59	63	67	71	73	77

Some of the values of n as published by the authors of the formula are as follows:

$n=0.010$ for well-planed timber or neat cement;
$n=0.012$ for common boards;
$n=0.013$ for ashlar or neatly joined brickwork;
$n=0.017$ for rubble masonry;
$n=0.020$ for canals in firm gravel;
$n=0.025$ for canals and rivers in good condition;
$n=0.030$ for canals and rivers with stones and weeds;
$n=0.035$ for canals and rivers in bad order.

The above values do not include all present construction materials, and later experimental data have shown the need of revising

HORTON'S VALUES OF THE COEFFICIENT OF ROUGHNESS, n, FOR KUTTER'S AND MANNING'S FORMULAS

Surface	Range of Values		Commonly used values
	From	To	
Vitrified sewer pipe	0.010	0.017	0.013
Common clay drain tile	0.011	0.017	0.014
Glazed brickwork	0.011	0.015	0.013
Brick in cement mortar	0.012	0.017	0.015
Neat-cement surfaces	0.010	0.013	
Cement-mortar surfaces	0.011	0.016	0.015
Concrete pipe	0.012	0.016	0.015
Plank flumes, planed	0.010	0.014	0.012
Plank flumes, unplaned	0.011	0.015	0.013
Plank flumes with battens	0.012	0.016	0.015
Concrete-lined channels	0.012	0.018	0.015
Rubble masonry	0.017	0.030	
Dry rubble	0.025	0.035	
Ashlar masonry	0.013	0.017	
Smooth metal flumes	0.011	0.015	
Corrugated metal flumes	0.022	0.030	
Earth canals in good condition	0.017	0.025	0.0225
Earth canals with weeds and rocks	0.025	0.040	0.035
Canals excavated in rock	0.025	0.035	0.033
Natural streams in good condition	0.025	0.033	
Natural streams with weeds and rocks	0.035	0.060	
Sluggish rivers, very weedy	0.050	0.150	

them. A more complete list of values of "Kutter's n," based upon later data and practice, and showing the probable ranges of variation has been prepared by Horton, extracts from which are tabulated on page 187.

The solution of the Kutter formula may be obtained from tables which usually accompany the formula. The use of the Chezy formula with the Kutter coefficient thus becomes much simplified. A short table of values of C corresponding to different values of r, s, and n is given on page 186. Interpolations are necessary in using this table.

118. The Manning Formula.—This formula was first suggested by Manning [1] in 1890. His study of the experimental data then available led to the conclusion that the values of the exponents y and z (formula (2)) which best represented the law of flow in open channels were, respectively $\frac{2}{3}$ and $\frac{1}{2}$. Expressed in English units the Manning formula is

$$v=\frac{1.486}{n}r^{2/3}s^{1/2}. \qquad (5)$$

This may be considered as the Chezy formula with

$$C=\frac{1.486}{n}r^{1/6}.$$

The coefficient of roughness, n, is to be given the same value as n in the Kutter formula. The values of n applicable to different channel conditions are tabulated on page 187. Expressed in metric units the Manning formula is

$$v=\frac{1}{n}r^{2/3}s^{1/2}. \qquad (6)$$

119. Comparison of Manning and Kutter Formulas.—Using the same value of n in each case, the Kutter and Manning formulas give identical results for $r=1$ meter$=3.28$ feet. This may be proved by substituting 3.28 for r in each formula. It will be found that each formula then reduces to

$$C=\frac{1.811}{n}.$$

[1] Robert Manning: Flow of Water in Open Channels and Pipes. *Trans. Inst. Civ. Eng. of Ireland*, 1890, vol. 20.

Since s does not appear in this equation, it follows that when r equals 1 meter, the Kutter formula gives the same value of C for all slopes.

Further investigation shows that for hydraulic radii less than 1 meter, with the same value of n used in each formula, the Kutter formula gives somewhat higher values of C than the Manning formula. For hydraulic radii greater than 1 meter the values of C obtained by the Kutter formula are in some cases slightly less and in others slightly greater than the values obtained by the Manning formula.

It has been found, however, from several hundred gagings of open channels which were made under a wide range of conditions as regards shape, size and variation in roughness that the proper values of n to be used in the two formulas are so nearly identical that for all practical purposes the same values may be used. In other words, with the same value of n, problems solved by means of one of the formulas will give results agreeing very closely with those obtained by using the other formula.

The Kutter formula has for many years been the most widely used of all of the open-channel formulas. It has been used almost exclusively in the United States and England and more commonly than any other formula in other parts of the world. The Manning formula has been used for a number of years, in Egypt, India, and Australia and quite recently many American engineers have come to see its advantages over the more cumbersome Kutter formula.

It is because of the established use of the Kutter formula and the general familiarity of engineers with the type of channel represented by different values of "Kutter's n" that there is an advantage in including n in the Manning coefficient. Expressed as it is, engineers familiar with the Kutter formula may adopt the Manning formula without the necessity of familiarizing themselves with a new coefficient, and at the same time get practically the same results as with the formula with which they are familiar.

Very evidently the Manning formula could be written

$$v = Kr^{2/3}s^{1/2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and values of K could be selected for different types of channels the same as values of n are now selected. It is to be hoped that this form of the formula will eventually come into general use, and

that the present form will be simply a step in the transition from the use of the Kutter formula to the use of the simplified form of the Manning formula. In order to assist in determining K in terms of n the following table of values is given:

MANNING'S K IN TERMS OF n. $K = \frac{1.486}{n}$

n	K	n	K	n	K	n	K	n	K	n	K
0.009	165	0.016	93	0.023	65	0.030	50	0.044	34	0.070	21
.010	149	.017	87	.024	62	.032	46	.046	32	.075	20
.011	135	.018	83	.025	59	.034	44	.048	31	.080	19
.012	124	.019	78	.026	57	.036	41	.050	30	.085	18
.013	114	.020	74	.027	55	.038	39	.055	27	.090	17
.014	106	.021	71	.028	53	.040	37	.060	25	.095	16
.015	99	.022	68	.029	51	.042	35	.065	23	.100	15

The Kutter formula shows C to be a function of the slope, s, while the Manning formula does not. The terms involving s in the Kutter formula were introduced to make the formula agree with the measurements of flow of the Mississippi river by Humphreys and Abbott. These measurements have since been shown to be in error by at least 10 per cent and to this extent the Kutter formula is based upon incorrect data. Later experiments do not verify the conclusions of Ganguillet and Kutter regarding this matter. The value of C in the Kutter formula is not materially affected by the s terms unless the slope is very small, much smaller than is ordinarily used in designing channels, and so the formula has been satisfactorily used for the conditions ordinarily encountered in practice. It is probable that the Kutter formula would have been more satisfactory for all channels, including those with very small slopes, with the "s" terms omitted. That this is so is shown quite conclusively by some recent experiments on flow in the Chicago drainage canal.[1]

The foregoing discussion may be summed up briefly with the statement that the Manning formula is much simpler to use than the Kutter formula and that, with the same value of n, it gives practically the same results as the Kutter formula except for flat

[1] MURRAY BLANCHARD: Hydraulics of the Chicago Sanitary Districts Main Channel. *Journal of the Western Society of Engineers*, Sept., 1920.

slopes. In the latter case the Manning formula appears to give more accurate results than the Kutter formula.

120. The Bazin Formula.—This formula, first published[1] in 1897, like the Kutter formula, determines the value of C in the Chezy formula. It considers C to be a function of r but not of s. Expressed in English units the formula is

$$C=\frac{157.6}{1+\frac{m}{\sqrt{r}}}, \qquad \qquad (8)$$

in which m is a coefficient of roughness. Values of m given by Bazin are

$m=0.109$ for smooth cement or planed wood;
$m=0.290$ for planks, ashlar and brick;
$m=0.833$ for rubble masonry;
$m=1.540$ for earth channels of very regular surface;
$m=2.360$ for ordinary earth channel;
$m=3.170$ for exceptionally rough channels.

The Bazin formula is used extensively in France but it has not been generally adopted in other countries. The value of m is subject to fully as wide a variation as n in the Kutter or Manning formulas.

121. Open-channel Formulas in General. — The Kutter, Manning, and Bazin formulas are the best known and most widely used of the open-channel formulas. There are a large number of other formulas which have been published, and many of these doubtless possess merit. It is not ordinarily advisable, however, to use any except the commonly accepted formulas unless there is very good reason for so doing. The successful use of any open-channel formula requires an accurate knowledge of conditions, and judgment in the selection of coefficients. Even the most experienced engineers may expect errors of at least 10 per cent in selecting coefficients with corresponding errors in their results.

122. Detailed Study of Hydraulic Gradient or Water Surface.—In the investigation of minor losses in pipes, Art. 102, it has been shown that these losses may be added collectively to the loss of head due to friction and that this sum, which represents the total

[1] Annales des Ponts et Chaussées, 1897.

lost head, may be considered as a unit. The losses of head for open channels are similar to those for pipes with the exception that they must be provided for at the places where they occur.

Losses of head in open channels which result from changes in velocity may be expressed as functions of the velocity head the same as for pipes. Thus,

$$h_0 = K_0 \frac{v^2}{2g}, \quad h_d = K_d \frac{v^2}{2g}, \quad h_c = K_c \frac{v^2}{2g}, \text{ etc.}$$

in which v is the mean velocity in the channel having the smaller cross-sectional area. While these losses of head for open channels are frequently of much greater importance than the similar losses for pipes, the values of coefficients for determining them are not so well established. More experimental data in this field are needed.

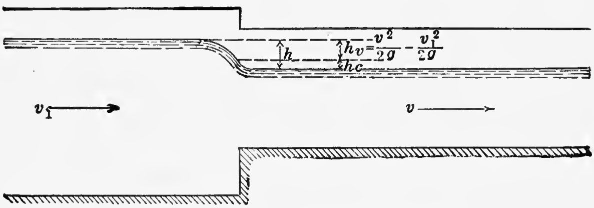

FIG. 112.—Change of channel to smaller section.

Fig. 112 shows the change in water surface resulting from *contracting the cross-sectional area* of a channel. The mean velocity in the larger channel is v_1 and in the smaller channel v. It is assumed that the grades of the two channels are just sufficient to maintain these velocities; that is, there is uniform flow in each channel.

Kinetic energy is always obtained at the expense of potential energy. In any open channel the drop in water surface, h in Fig. 112, occurring at any change in section measures the loss in potential energy resulting from the change. A portion of h, h_v in figure, is used in producing kinetic energy, that is in increasing the velocity of the water. The remainder, h_c, is the head used in overcoming friction losses at the place where the change in velocity occurs. Referring to the figure,

$$h_v = \frac{v^2}{2g} - \frac{v_1^2}{2g}. \qquad (9)$$

in which v and v_1 are the mean velocities in the smaller and larger channels respectively. This expression is also obtained by writing Bernoulli's equation between points in a filament on either side of the change in section when equal velocities at all points in a cross-section are assumed for each channel.

From the figure

$$h=h_c+h_v=h_c+\frac{v^2}{2g}-\frac{v_1^2}{2g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and in a manner similar to that employed for losses due to contraction in pipes

$$h_c=K_c\frac{v^2}{2g} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and substituting this value in equation (10)

$$h=K_c\frac{v^2}{2g}+\frac{v^2}{2g}-\frac{v_1^2}{2g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

There are no experimental data from which K_c for open channels can be determined but it appears reasonable that it may have a value corresponding to that for contractions in pipes. The maximum value for a sharp-cornered entrance may thus be taken as 0.5 with a smaller value for a rounded or tapered entrance. With care in design the value of K_c may be reduced very nearly to zero. Under the most favorable conditions where K_c is zero, the difference in elevation of water surfaces will be h_v. This drop in water surface should always be provided for when a canal changes to a smaller section.

Example.—Assume entrance conditions, such that $K_c=0.25$; the velocity in the upper channel, $v_1=2.0$ ft. per second; in the lower channel $v=8.0$ ft. per second. Determine the drop in water surface. From equation (12)

$$h=0.25\times\frac{8^2}{64.32}+\frac{8^2}{64.32}-\frac{2^2}{64.32}=1.18 \text{ ft.}$$

If a canal discharges from a reservoir or other body of comparatively still water the conditions are the same as above except that v_1 may be considered zero. In the above problem v_1 could have been considered zero without materially affecting the result.

Fig. 113 shows the change in water surface resulting from *enlarging the cross-sectional area of a channel.* The mean velocity in the

smaller channel is v and in the larger channel v_1. It is assumed that the slopes of the two channels are just sufficient to maintain these velocities.

The velocity v being greater than v_1 there is a loss in kinetic energy with a resultant gain in potential energy. If there were no loss of energy from friction, all of the kinetic energy in the smaller channel in excess of the kinetic energy in the larger channel would be converted into potential energy, and the water surface in the larger channel would be at a distance h_v above the water surface in the smaller channel. Since some energy is required to overcome friction and turbulence losses where the transition in velocities occurs, the actual elevation of water surface in the larger channel is lower than it would be if there were no friction losses, or, as indicated in the figure, at a distance, h, above the water surface in the smaller channel. The vertical distance $h_e = h_v - h$ thus rep-

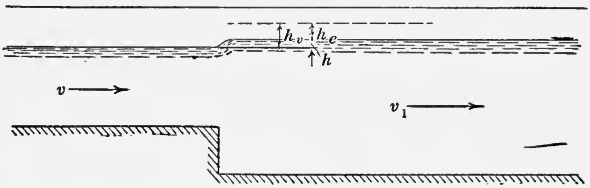

FIG. 113.—Change of channel to larger section.

resents the loss of head due to enlargement. This may be expressed algebraically as follows:

$$h_v = \frac{v^2}{2g} - \frac{v_1^2}{2g}; \quad \quad (13)$$

also

$$h = h_v - h_e = \frac{v^2}{2g} - \frac{v_1^2}{2g} - h_e, \quad \quad (14)$$

and substituting $K_e \frac{v^2}{2g}$ for h_e the formula may be written,

$$h = \frac{v^2}{2g} - K_e \frac{v^2}{2g} - \frac{v_1^2}{2g}. \quad \quad (15)$$

There are no satisfactory experimental data giving values of K_e. In general, however, it is known that for abrupt changes in velocity, very little of the kinetic energy in the smaller channel is converted into potential energy. In other words nearly all of this energy is lost in friction and turbulence and there is little differ-

ence in elevation in the water surfaces in the two channels. In this case K_e approaches very near to unity. By exercising care in design and construction and making velocity changes gradual so as to produce a minimum of turbulence the value of K_e may be greatly reduced.

It should be remembered that equations (9) to (15) are approximations in that they assume the kinetic energy in each channel to be that due to the mean velocity, that is, they assume equal velocities at all points in a cross-section for each channel. This is equivalent to assuming α in equation (1) to be equal to unity. The error introduced, thereby is however, of no great importance, especially in view of the fact that a correction for this error is necessarily included in the empirical values selected for K_c or K_e.

The most common types of *obstructions in open channels* are submerged weirs, gates, and bridge piers. Losses of head resulting from weirs of all kinds and gates, which are types of orifices, are treated in earlier chapters. These structures are commonly employed to deflect water from main canals to secondary channels.

Bridge piers, Fig. 114, restrict the cross-sectional area of a channel and therefore obstruct the flow. The loss of head, h_g, or what amounts to the same thing, the amount that the water will be backed up by piers is not sufficient to be of any importance except for comparatively high velocities. The most important case arises in determining the backing-up effect of bridge piers during flood stages of streams. The total loss of head is made up of three parts; a loss of head due to contraction of the channel at the upstream end of the piers, a loss of head due to enlargement of the channel at the downstream end, and an increase in loss of head due to friction resulting from the increase in velocity in the contracted portion of the channel. On account of the higher velocity, the surface of the water between the piers is depressed, the vertical distance, h, measuring the increase in velocity head plus the loss of head. The distance $h-h_g$ is a measure of the velocity head reconverted into static head.

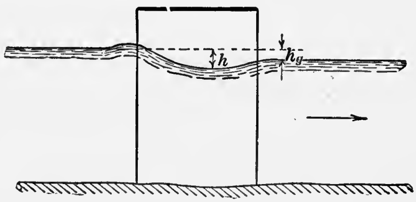

FIG. 114.—Bridge pier obstructing flow.

The quantitative determination of losses of head from piers is entirely empirical. Though many experimental data are available, no satisfactory general formula for the solution of this problem is known. In general, however, piers that are so designed as to allow the changes in velocity to occur gradually with a minimum amount of turbulence, cause the smallest loss of head. Fig. 115 represents two horizontal sections of piers. Section A will cause less turbulence and consequently less loss of head than section B.

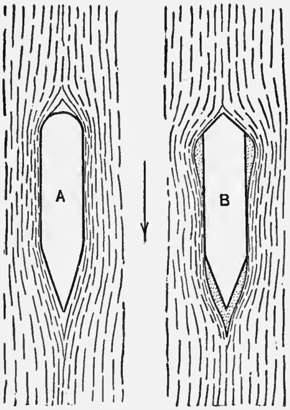

Fig. 115.—Effect of shape of bridge piers in causing turbulence.

Curves or bends in the alignment of a channel cause a loss of head. For low velocities such as occur in earth canals the loss in head is slight and ordinarily no allowance is made for it unless the curves are sharp and frequent. For sharp curves in concrete-lined canals or flumes designed to carry water at high velocities, an increase in the slope should be provided. There are few experiments for determining the loss of head in curves, but the data for bends in pipes (page 163) may be used as a guide. Some engineers prefer to correct for loss of head at curves by using a higher coefficient of roughness.

123. Hydraulics of Rivers.—Open-channel formulas do not apply accurately to natural streams since the channel sections and slope of water surface vary and the flow is non-uniform. At the lower stages, streams usually contain alternating reaches of riffles and slack water. During high stages this condition largely disappears and the water surface becomes approximately parallel to the average slope of the bottom of the channel. The degree of roughness of natural streams varies greatly within short reaches and even within different parts of the same cross-section. This may be seen from Fig. 116, which illustrates a stream in flood stage. The channel of normal flow, *abc*, will probably have an entirely different coefficient of roughness than the flood plain *cde*. Also the portion of the left-hand bank, Fig. 116, lying above ordinary high water may be covered with trees or other vegetation and have a

higher coefficient than the lower portion. Rocks and other channel irregularities are frequent and cause varying conditions of turbulence, the effect of which on the coefficient of roughness is difficult to estimate.

There are, however, times when the engineer must estimate as well as he can, the carrying capacity of a natural channel. This is done by making a survey of the stream, from which cross-sections may be plotted and the slope of water surface may be determined. A certain reach is selected for which are obtained an average cross-section and slope of water surface, the computations being based upon these average values.

It is apparent that results obtained in this manner will be very approximate, and that the degree of accuracy obtained will depend largely upon the ability of the engineer to judge the effect of these varying conditions upon the coefficient of roughness. Better results will be obtained for natural streams that have fairly straight and uniform channels and are free from conditions causing turbulence. At the higher stages channel irregularities have less effect upon the slope of water surface, and open-channel formulas then apply more accurately.

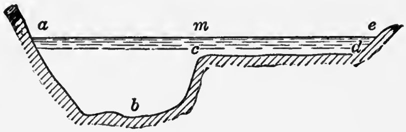

Fig. 116.—Natural stream in flood stage.

124. Irregular Sections.—Open-channel formulas should not be applied directly to sections having a break or pronounced irregularity in the wetted perimeter. Fig. 116, which illustrates a stream in flood stage shows a break in the wetted perimeter at c. That the open-channel formulas do not apply directly to the entire cross-section in such a case may be shown by the following example. The Manning formula is used, though any of the other open-channel formulas will show substantially the same results. Assume $s=0.001$; $n=0.035$; the length of the portion, abc, of the wetted perimeter $=200$ ft., and of the portion $cde=300$ ft.; the area of the portion, $abcm$ of the cross-section $=$ 3000 sq. ft. and of the portion $mcde=900$ sq. ft. Then for the entire cross-section, $a=3900$, $p=500$, $r=\frac{a}{p}=\frac{3900}{500}=7.8$, and by the Manning formula, $Q=20{,}600$ cu. ft. per second. The cross-section may now be divided into the two portions, $abcm$ and

mcde. Assuming the depth of water over the flood plain to be 3 ft., the wetted perimeter of the portion of the cross-section, *abcm* will be about 203 ft. Then $r=\frac{a}{p}=\frac{3000}{203}=14.78$ and from the Manning formula, $Q=24{,}250$ cu. ft. per second. This indicates that a portion of the channel discharges more water than all of it, which is clearly impossible.

Where an open-channel formula must be applied to an irregular section such as that indicated in Fig. 116, it is necessary to divide the cross-section into two portions and compute the discharges for each portion separately. As the two portions of the channel will differ in roughness, different coefficients should be selected for each.

125. Cross-section of Greatest Efficiency.—The most efficient channel cross-section, from a hydraulic standpoint, is the one which, with a given slope and area, will have the maximum capacity. This cross-section is the one having the smallest wetted perimeter, since frictional resistance increases directly with the wetted perimeter. This also may be seen from an examination of one of the open-channel formulas. Take for example the equation for Q as given by the Manning formula

$$Q=av=\frac{1.486}{n}ar^{2/3}s^{1/2}. \qquad (16)$$

Under the assumptions made a, n, and s are constant. Q therefore increases with r. Since $r=\frac{a}{p}$, r increases as p decreases and since Q varies only with r it is a maximum when p is a minimum.

It should be borne in mind in this connection that there are usually practical objections to using cross-sections of minimum area but the dimensions of such cross-sections should be known and adhered to as closely as conditions appear to justify.

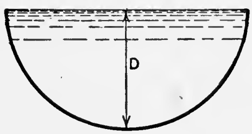

Fig. 117.—Semicircular channel.

Of all cross-sections, having a given area the semicircle, Fig. 117, has the smallest wetted perimeter and it is therefore the cross-section of highest hydraulic efficiency. Semicircular cross-sections are sometimes used for concrete or brick channels but they are not used for earth channels.

Trapezoidal cross-sections, Fig. 118, are used more commonly than any others. They are the only practical sections for earth canals, and masonry and wooden conduits are usually of this form. The rectangular section, usually used for wooden flumes, may be considered as a special case of the trapezoidal section. Properties of trapezoidal sections and methods of determining sections of greatest efficiency are shown in the following analysis.

From Fig. 118, $\frac{e}{D}=z$ and $\frac{b}{D}=y$; or $e=Dz$ and $b=Dy$. Then

$$p=(y+2\sqrt{1+z^2})D, \qquad (17)$$

and

$$a=D^2(z+y), \qquad (18)$$

or

$$D=\sqrt{\frac{a}{z+y}}. \qquad (19)$$

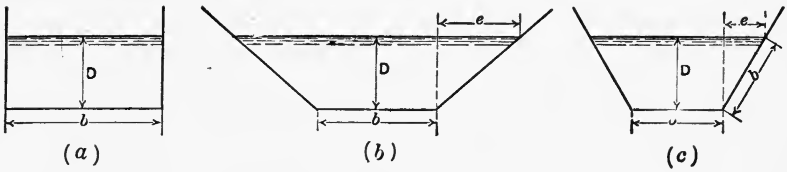

FIG. 118.—Trapezoidal channels.

Substituting this value of D in equation (17),

$$p=(y+2\sqrt{1+z^2})\sqrt{\frac{a}{z+y}}. \qquad (20)$$

Equating the first derivative with respect to y to zero and reducing,

$$y=2(\sqrt{1+z^2}-z), \qquad (21)$$

or

$$b=2D(\sqrt{1+z^2}-z). \qquad (22)$$

From which may be obtained the relation between depth of water and bottom width of canal of the most efficient cross-section for any values of z.

From equations (17) and (18),

$$\frac{a}{p}=r=\frac{D^2(z+y)}{D(y+2\sqrt{1+z^2})}. \qquad (23)$$

Substituting y from equation (21) and reducing

$$r=D/2, \qquad (24)$$

or, the cross-section of greatest efficiency has a hydraulic radius equal to one-half the depth of water.

By substituting y from equation (21) in equation (20) and reducing, the following expression is obtained

$$p=2\sqrt{a}\sqrt{2\sqrt{1+z^2}-z}, \qquad (25)$$

equating the first derivative with respect to z to zero and reducing

$$z=\frac{1}{\sqrt{3}}=\tan 30°.$$

It may be seen from an examination of equation (22) that when $z=\tan 30°$ the length of each side is equal to b, Fig. 118 (c) and the section becomes a half hexagon. Thus, of all the trapezoidal sections (including the rectangle), for a given area, the half hexagon has the smallest perimeter and it is therefore the most efficient trapezoidal cross-section.

From equation (22) are obtained the following relations between bottom width and depth (for different side slopes) for trapezoidal cross-sections of maximum efficiency.

$e/D=z$	0	$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	3	4
b	$2D$	$1.56D$	$1.24D$	$0.83D$	$0.61D$	$0.47D$	$0.32D$	$0.25D$

A semicircle having its center in the middle of the water surface may always be inscribed within a cross-section of maximum efficiency. This is illustrated for a trapezoidal cross-section in Fig. 119. OA, OB, and OC are drawn from a point O on the center line of the water surface perpendicular to the sides of the channel EF, FG, and GH respectively. Let $EF=GH=x$; $FG=b$; $OA=OC=R$; and $OB=D$. As before a=area of section and p=wetted perimeter. Then from the figure

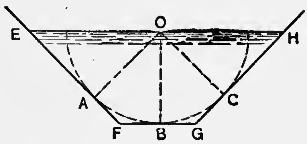

Fig. 119.

$$a=xR+\tfrac{1}{2}bD$$

and

$$p=2x+b.$$

And since (equation (24)) the hydraulic radius equals one-half the depth of water

$$\frac{a}{p}=\frac{xR+\frac{1}{2}bD}{2x+b}=\frac{D}{2}.$$

From which

$$R=D.$$

That is, OA, OB, and OC are all equal and a semicircle with center at O is tangent to the three sides.

126. Circular Sections.—The maximum discharge from a channel of circular cross-section occurs at a little less than full depth. This may be seen from an examination of open-channel formulas. The discharge by the Manning formula is

$$Q=\frac{1.486}{n}ar^{2/3}s^{1/2}, \quad \ldots \ldots \ldots \quad (16)$$

a being the cross-sectional area. In the investigation of a particular channel n and s will be constant. From Fig. 120, R being the radius of the circle,

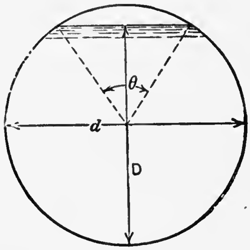

FIG. 120.—Circular channel.

$$p=\frac{360-\theta}{360}\times 2\pi R$$

and

$$a=\frac{360-\theta}{360}\times\pi R^2+\tfrac{1}{2}R^2\sin\theta.$$

With these equations an expression for $ar^{2/3}$ may be written, differentiating which and equating to zero, the value of θ which makes Q a maximum is found to be 57° 40′. The corresponding depth of water is $D=0.938d$. Other open-channel formulas will give substantially the same result. This means that a pipe carrying water not under pressure, when free from obstructions and laid on a true grade, will not flow full.

127. Non-uniform Flow.—In uniform flow the velocity past all cross-sections in the channel is constant. This condition obtains in ordinary conduits where successive cross-sections are uniform in size and shape and the slope of water surface is parallel to the bed of the channel. There are, however, certain cases where the velocities are being accelerated or retarded, that is, the flow is non-uniform, although the same quantity of water passes all

cross-sections so that continuity of flow exists, or expressed by symbols,

$$Q = a_1 v_1 = a_2 v_2 = a_3 v_3, \text{ etc.}$$

An example of non-uniform flow is illustrated in Fig. 121, which represents a canal supplied by another canal or reservoir. Water enters the canal at a certain initial velocity which may be computed by the method described in Art. 85. There is a sudden drop in the water surface at the entrance. The slope of the canal is greater than that required to carry the water at its initial velocity and the velocity therefore continues to accelerate until it becomes equal to the velocity at which the channel will carry water under conditions of uniform flow.

Fig. 122 which represents a canal connecting two reservoirs is

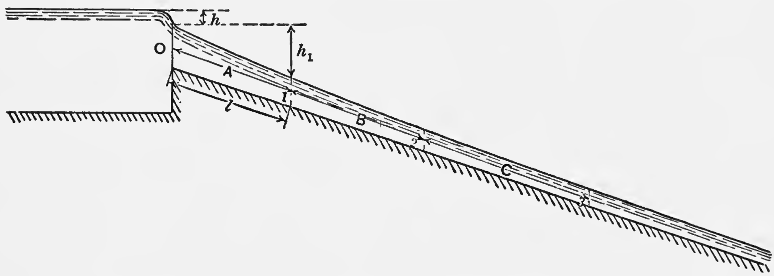

FIG. 121.—Non-uniform flow in channel with steep slope.

another example of non-uniform flow. The bottom of the canal is on a slope different from that which the water surface will attain. The total head producing flow is H. As before, there is a drop in the water surface at the place where the water receives its initial velocity. The same analysis applies to the two cases illustrated in Figs. 121 and 122.

This problem may best be analyzed by considering the channel to be divided up into reaches A, B, C, etc., Figs. 121 and 122, the computations being made for one reach at a time. There is a certain degree of approximation introduced in doing this as computations are based upon an average cross-sectional area, but by reducing the length of reach considered any desired degree of accuracy may be obtained. The following nomenclature is used:

l = length of reach considered;
s_1 = slope of bottom of canal;
s = average slope of water surface in reach;
h_1 = fall of water surface in reach;
h_f = loss of lead due to friction in reach;
d_0 = depth of water in upper end of reach;
d_1 = depth of water in lower end of reach;
b_0 = bottom width of trapezoidal section at upper end of reach;
b_1 = bottom width of trapezoidal section at lower end of reach;
v_0 = mean velocity at upper end of reach;
v_1 = mean velocity at lower end of reach;
v = mean velocity at middle of reach;
r = hydraulic radius of section at middle of reach;
z = slope of sides of canal for trapezoidal section;
n = coefficient of roughness in Manning's formula.

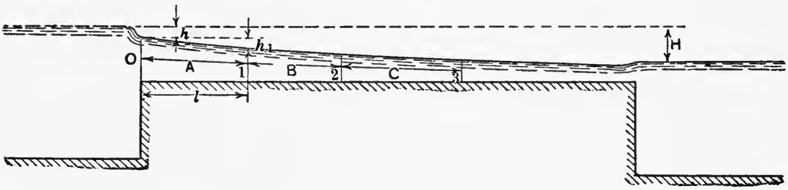

Fig. 122.—Non-uniform flow in channel with flat slope.

From Bernoulli's equation, assuming equal velocities at all points in a cross-section.

$$h_1+\frac{v_0^2}{2g}=h_f+\frac{v_1^2}{2g}. \qquad (26)$$

From Manning's formula,

$$v=\frac{1.486}{n}r^{2/3}s^{1/2}=\frac{1.486}{n}r^{2/3}\left(\frac{h_f}{l}\right)^{1/2}, \qquad (27)$$

and approximately, putting $v=\frac{1}{2}(v_0+v_1)$,

$$h_f=\frac{ln^2(v_0+v_1)^2}{8.83r^{4/3}}. \qquad (28)$$

Substituting this value of h_f in equation (26) and transposing,

$$h_1=\frac{v_1^2}{2g}-\frac{v_0^2}{2g}+\frac{ln^2(v_0+v_1)^2}{8.83r^{4/3}}. \qquad (29)$$

This is the general equation for non-uniform flow.

If the areas and wetted perimeters of the cross-sections at the upper and lower ends of a reach and also the drop in water surface, h_1, are measured, the velocity at one end of the reach may be expressed in terms of the velocity at the other end, as, for example, $v_0=\frac{a_1v_1}{a_0}$, and the other velocity, v_1, may be computed from equation (29).

Similarly if Q and the cross-section at the upper end of a reach are known, the cross-section at the lower end of a reach may be computed. Assume for example a trapezoidal cross-section. Then

$$h_1=d_0+s_1l-d_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

$$v_0=\frac{Q}{d_0(b_0+zd_0)}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

$$v_1=\frac{Q}{d_1(b_1+zd_1)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

and for the average section,

$$r=\frac{d_0(b_0+zd_0)+d_1(b_1+zd_1)}{b_0+b_1+2(d_0+d_1)\sqrt{1+z^2}}. \quad . \quad . \quad . \quad . \quad (33)$$

In the right-hand members of the above equations b_1 and d_1 are the only unknown quantities, and one of these must be assumed. The known quantities and the assumed value of b_1 or d_1 are then substituted in these equations and the expressions for h_1, v_0, v_1, and r thus obtained are substituted in equation (29) which equation may be solved for b_1 or d_1 depending upon which has been assumed. Suppose for example that d_1, the depth of water at the lower end of the reach, has been assumed; b_1 is then computed. If the proportions of canal section thus obtained are not satisfactory, a new value of d_1 may be assumed and b_1 may be recomputed. Ordinarily the general form of the canal will be well enough known so that recomputations will be unnecessary.

To get the dimensions of other cross-sections the above process will be repeated, the section at the lower end of one reach becoming the upper cross-section of the next reach below.

128. Backwater.—A common problem in non-uniform flow occurs where water is backed up by a dam, weir or other obstruction. Usually it is required to determine the amount that the

water surface will be raised at certain specified distances upstream from the obstruction. Fig. 123 indicates a channel whose water surface without the obstruction would be the line *mn*; with the obstruction the water surface assumes the curved line *abcde*. This latter line is called the *backwater curve.* It is required to determine the position of sufficient points on the backwater curve so that it may be plotted.

In this case the velocity is retarded but the general principles are the same as for non-uniform flow with accelerating velocity, and the same general method of solution may be followed. The channel is divided up into reaches A, B, C, etc., as before, but as the elevation of water surface at the obstruction is usually given the computations are begun at this section and continued upstream.

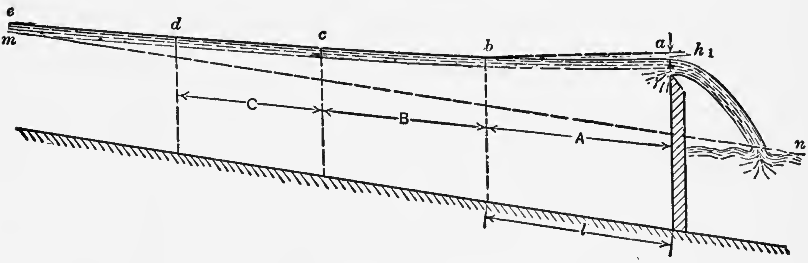

Fig. 123.—Backwater.

This is not necessary, however, as the computations may be begun at any section with a known elevation of water surface and be carried either upstream or downstream. The nomenclature is that given on page 203.

The general equation for non-uniform flow as given on page 203 is

$$h_1 = \frac{v_1^2}{2g} - \frac{v_0^2}{2g} + \frac{ln^2(v_0+v_1)^2}{8.83r^{4/3}}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (29)$$

d_0, the depth of water at the upper end of the reach is the quantity sought. With this determined the depth at the upper end of the next reach may be obtained in the same manner. This problem may be solved by equation (29) in a manner practically identical with that described in Art. 126, the only difference being that the computations proceed upstream instead of downstream.

One of the commonest applications of the backwater curve

problem is to the determination of the elevations of points in the backwater curve above a dam. This is necessary when the damage which will result from submerging property during floods or the effect of backwater on some power plant farther upstream, is to be determined. Since in natural streams the channels are irregular, average sections in each reach and also average velocities are used. It is usual, therefore, to put the average velocity in the reach, v, in formula (29) in the place of v_0 and v_1, from which the following simplified expression is obtained:

$$h_1=\frac{ln^2v^2}{2.21r^{4/3}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

This is simply a transposed form of the Manning formula, h_1 in this case being equal to h_f.

In solving backwater problems by this formula, Q and the elevation of water surface at the lower end of reach A are usually known. Then h_1 is assumed, from which a trial value of elevation of water surface in the middle of the reach may be obtained. Based upon this assumed elevation an average channel cross-section for the reach may be plotted and the area and hydraulic radius determined. Then h_1 may be computed and if this computed value of h_1 differs from the assumed value sufficiently to materially affect the results of the computations, a new assumption for h_1 may be made and the computations repeated until the assumed value of h_1 is as near as is desired to the assumed value. With a little experience it will be found that the first assumed value of h_1 will give the computed value close enough without repeating the computations.

As considerable uncertainty exists in the selection of a proper value of n and since the error thus introduced into the result is raised to the second power any solution of this problem is necessarily approximate.

129. Divided Flow.—Fig. 124 represents a channel divided by an island. The total discharge, Q, is given and it is required to determine Q_1 and Q_2, the portion of discharge going respectively to channels 1 and 2, and also the total lost head h_1, that is, the drop in water surface from m to n. From the Manning formula

$$s=\frac{h_1}{l}=\frac{n^2v^2}{2.21r^{4/3}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

and since $v=\frac{Q}{a}$, using subscripts 1 and 2 to refer to the respective channels,

$$\frac{h_1}{l_1}=\frac{n_1^2\left(\frac{Q_1}{a_1}\right)^2}{2.21r_1^{4/3}}, \quad \ldots \ldots \ldots \quad (36)$$

and

$$\frac{h_1}{l_2}=\frac{n_2^2\left(\frac{Q_2}{a_2}\right)^2}{2.21r_2^{4/3}}. \quad \ldots \ldots \ldots \quad (37)$$

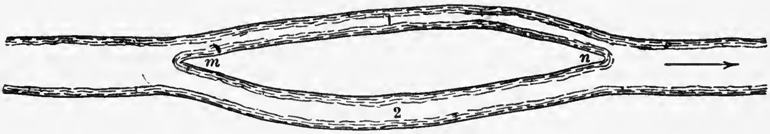

FIG. 124.—Channel divided by island.

Equating values of h_1 and reducing

$$Q_1=Q_2\frac{n_2a_1r_1^{2/3}}{n_1a_2r_2^{2/3}}\sqrt{\frac{l_2}{l_1}}. \quad \ldots \ldots \ldots \quad (38)$$

Putting

$$F=\frac{n_2a_1r_1^{2/3}}{n_1a_2r_2^{2/3}}\sqrt{\frac{l_2}{l_1}}, \quad \ldots \ldots \ldots \quad (39)$$

$$Q_1=FQ_2, \quad \ldots \ldots \ldots \quad (40)$$

and since

$$Q=Q_1+Q_2, \quad \ldots \ldots \ldots \quad (41)$$

$$Q_2=\frac{Q}{1+F}, \quad \ldots \ldots \ldots \quad (42)$$

h_1 may now be obtained by substituting Q_1 and Q_2 in formulas (36) and (37).

In computing backwater curves where the channel is divided by an island of considerable length it may be necessary to make computations for separate reaches, as described on page 205. In this case, Q_1 and Q_2 may be determined approximately from formulas (40) and (42) using an average cross-section for each channel and with these discharges determined the slope, or rise in water surface, h_1, may be computed. If the values Q_1 and Q_2 are correct, the computations should show h_1 the same for each

channel. If the computed values of h_1 do not agree, the computations should be repeated, reducing the discharge of the channel for which the computations give the greatest value of h_1 and increasing the discharge of the other channel by an equal amount. The computations are repeated until they give approximately the same values of h_1 for both channels.

It may be helpful, in order to reduce the number of trial solutions, to plot values of Q against the error made in each assumption. The method is similar to that for divided flow in pipes, described on page 167.

The problem presented by channels having a flood plain, Fig. 116 is similar to that of the channel divided by an island. The flow over the plain should be computed separately from that for the main channel, as discussed on page 207. If the total discharge is known, Q_1 and Q_2 being respectively the portions of the flow for the main channel and flood plain, formulas (40) and (42) may be used to determine approximate values of Q_1 and Q_2. With these approximate discharges determined, the two values of h_1 may be computed, using the proper value of n for each channel. If these values of h_1 do not agree the computations should be repeated, correcting Q_1 and Q_2 and continuing the computations in the same manner as that described above for channels divided by an island.

PROBLEMS

1. An earth canal in good condition having a bottom width of 12 ft. and side slopes of 2 horizontal to 1 vertical is designed to carry 180 cu. ft. per second at a mean velocity of 2.25 ft. per second. What is the necessary grade of the canal?

2. What is the capacity of the canal in Problem 1, if the grade is 2 ft. per mile, all other conditions remaining the same?

3. Determine the depth of water in the canal, described in Problem 1, if the grade is 2 ft. per mile, other conditions remaining as stated.

4. Determine the bottom width of the canal having the same capacity and side slopes as the canal described in Problem 1, that will give the most efficient section.

5. A circular concrete sewer 5 ft. in diameter and flowing half full has a grade of 4 ft. per mile. Determine the discharge.

6. In Problem 5, what slope in feet per mile must the sewer have if the mean velocity is to be 8 ft. per second when flowing full capacity?

7. A smooth-metal flume of semicircular cross-section, has a diameter of 6 ft. and a grade of 0.005. What diameter of corrugated metal flume will be required to have the same capacity.

8. An earth canal carries a depth of water of 6 ft. The canal is 20 ft. wide on the bottom and has side slopes of 1.5 horizontal to 1 vertical. $s = 0.0002$. Using a value of n of 0.025 compute the discharge by the Manning formula and with this discharge determine the value of n in the Kutter formula and also the value of m in the Bazin formula.

9. An earth canal is to be designed to carry 400 cu. ft. per second at a mean velocity of 2.2 ft. per second. The sides of canal have a slope of 2 horizontal to 1 vertical. The depth of water is to be one-fourth of the bottom width. Assuming that the canal will be maintained in good condition find the necessary grade.

10. An earth canal in good condition carries 200 cu. ft. per second at a velocity of 2 ft. per second. Side slopes of canal are 2 horizontal to 1 vertical. The depth of water is one-third of bottom width of canal. This canal discharges into a flume with a tapered entrance, the conditions being such that the loss of head at entrance may be considered to be one-half of what it would be for an abrupt change in section. The flume is 7 ft. wide and has vertical sides. The slope of the bottom of the flume is such that it carries a depth of water of 3.5 ft. Determine how much the bottom of the flume should be above or below the bottom of the canal.

11. An earth canal containing weeds and grass has a bottom width of 15 ft. and side slopes of 2 horizontal to 1 vertical. The depth of water is 4 ft. and the slope is 2.75 ft. per mile. It is desired to change the section to a semicircular concrete-lined channel having a slope of 1.5 ft. in 1000 ft. Determine the radius of the semicircular channel if it flows full. If the change in section is abrupt and sharp-cornered, what will be the drop in water surface where the change in section occurs?

12. An earth canal in good condition has a bottom width of 10 ft., side slopes of 1.75 horizontal to 1 vertical, a grade of 0.00025, and carries 140 cu. ft. per second. A gate is constructed at the lower end of the canal which discharges freely into the air. The coefficient of discharge of the gate is 0.83. The gate is to be 3.0 ft. high. How wide should the gate be to maintain a constant depth of water in the canal?

13. A rectangular flume of unplaned timber connects two reservoirs 300 ft. apart. The flume is 16 ft. wide and both entrance and exit are sharp-cornered. The bottom of the flume, which is on a level grade, is 5 ft. below the water surface in the upper reservoir and 2 ft. below the level in the lower reservoir. Determine the discharge.

14. A rectangular flume of unplaned timber carries water from a reservoir. The width of flume is 20 ft., the length is 1000 ft. and the slope is 1 ft. per 100 ft. If the entrance is sharp-cornered and the bottom of the flume at the entrance is 4 ft. lower than the water surface in the reservoir, determine the rate of discharge.

15. An earth canal in good condition is 60 ft. wide on the bottom and has side slopes of 2 horizontal to 1 vertical. One side slope extends to an elevation of 20 ft. above the bottom of the canal. The other bank, which is a practically level meadow at an elevation of 6 ft. above the bottom of the canal, extends back 500 ft. from the canal and then rises abruptly. The meadow is covered with short grass and weeds. If the slope of the canal

is 2.2 ft. per mile, determine the discharge when the water is 8 ft. deep in the canal.

16. A flume built of planed lumber, with vertical sides, 8 ft. wide has a grade of 2 ft. per 1000 ft. A sharp-crested weir 3.5 ft. high is constructed across the flume. When the head of water over the weir is 3.1 ft., what is the elevation of water surface at a section 200 ft. upstream from the weir? How much higher is the water surface at this section than it would be with the same quantity of water flowing but with the weir removed?

17. A canal carries 300 cu. ft. per second of water at a depth of 5.5 ft. The water in this canal is to be dropped to a lower elevation through a concrete chute of rectangular cross-section, having a grade of 1 ft. in 10 ft. The chute is to carry a uniform depth of water of 3.0 ft., the width to vary as required to maintain this depth. Determine the width of chute at entrance, at 100 ft. below the entrance, and at 200 ft. below the entrance. Also determine the minimum width possible for this depth of water.

18. A canal, 58,000 ft. long (580 Sta.'s), is to be constructed with a capacity of 300 cu. ft. per second. The canal diverts from a river and terminates at a reservoir into which it discharges. The water surface in the river at the point of diversion is to be maintained at an elevation of 770 ft.

(*a*) Water is to be diverted through six head gates, having rectangular orifices each 2 ft. by 5 ft. Determine the head required to force the water through these openings, assuming a coefficient of discharge of 0.80.

(*b*) From Sta. 0 to Sta. 425 the canal is in earth section, having side slopes of 2 horizontal to 1 vertical, and a depth of water of 0.3 of the bottom width of the canal. Velocity of water is to be 2.1 ft. per second. Assume a coefficient of roughness of 0.0225. Determine grade or slope of canal.

(*c*) Between Sta. 425 and 500 the canal is in rock and is to have a semicircular section lined with concrete. The grade of the canal is to be 2 ft. per 1000 ft. Coefficient of roughness, 0.014. Determine the head lost at entrance, using a coefficient of velocity of 0.92. Also determine diameter of canal section.

(*d*) From Sta. 500 to Sta. 580 the section of canal is the same as from Sta. 0 to Sta. 425. At the reservoir end of the canal (Sta. 580) a weir is to be constructed in order that a uniform depth of water may be maintained throughout the entire length of earth section. Length of this weir is to be equal to the bottom width of the earth canal. The weir has a rectangular section, with horizontal crest 2 ft. 8 in. wide. Determine height of crest above bottom of the canal.

(*e*) Tabulate the elevations of the water surface, to nearest 0.1 ft., at the following stations: Sta. 0+10, Sta. 424+90, Sta. 425+10, Sta. 500 and Sta. 579+90. (Assume K_e-1 at Sta. 500.)

CHAPTER XI

HYDRODYNAMICS

130. Fundamental Principles.—Newton's laws of motion form the basic principles of the subject of hydrodynamics. These laws are clear and definite and lead to results that agree exactly with experiment. Briefly stated they are as follows:

I. Any body at rest or in motion with a uniform velocity along a straight line will continue in that same condition of rest or motion until acted upon by some external force.

II. Any change in the momentum of a moving body is proportional to the force producing that change and occurs along the same straight line in which the force acts.

III. To every action there is always an equal and opposite reaction.

These three laws of Newton's are frequently referred to as the Laws of Inertia, Force, and Stress, respectively. The solution of practically any problem in hydrodynamics may be accomplished by the direct application of these laws. It is therefore essential that a clear conception be had of their full significance. As an aid in acquiring this conception the following discussion is presented.

131. Interpretation of Newton's Laws.—Newton's first law of motion is merely a statement of the now well-known fact that matter is inert; that is, it possesses no ability, *per se*, to change its condition of rest, or motion, and that any such change must be brought about through the action of some external force, as for instance, friction of the air in retarding the velocity of a bullet.

Since change in motion results from the application of a force it may be assumed that the magnitude of the change in motion will depend upon the magnitude of the force producing that change; in other words it may be assumed that there is, as usual, a direct relation between cause and effect. Making this assumption, the second law follows naturally from the first. Momentum is by definition *quantity of motion*, and is equal to the product of

the mass and velocity. Designating by Z_1, the momentum of a mass of water, M, having a velocity v_1.

$$Z_1 = Mv_1.$$

If a force P acting upon this mass for a time Δt changes the velocity to v_2.

$$Z_2 = Mv_2,$$

and

$$Z_2 - Z_1 = M(v_2 - v_1), \quad \ldots \quad (1)$$

or

$$\Delta Z = M\Delta v,$$

and

$$\frac{\Delta Z}{\Delta t} = M\frac{\Delta v}{\Delta t} = P, \quad \ldots \quad (2)$$

since $\frac{\Delta v}{\Delta t}$ = acceleration, and mass times acceleration equals force.

Equation (2) may be written in the form,

$$P = \frac{M}{\Delta t}\Delta v, \quad \ldots \quad (3)$$

or substituting for Δv its equivalent $(v_2 - v_1)$ and letting Δt equal 1 second, equation (3) becomes,

$$P = M(v_2 - v_1). \quad \ldots \quad (4)$$

This means that when a force acts upon a mass and thereby changes its velocity from v_1 to v_2, the force is equal to the product of the mass whose velocity is changed each second from v_1 to v_2, and the change in velocity.

This may be demonstrated in another manner. The amount of work done upon any mass is equal to the gain in kinetic energy, or

$$Pl = \frac{Mv_2^2}{2} - \frac{Mv_1^2}{2} = \frac{M}{2}(v_2 - v_1)(v_2 + v_1), \quad \ldots \quad (5)$$

in which l is the distance through which the force P acts upon the mass M. But

$$l = \left(\frac{v_2 + v_1}{2}\right)t. \quad \ldots \quad (6)$$

Therefore, from equations (1) and (2)

$$Pt = M(v_2 - v_1), \quad \ldots \quad (7)$$

or letting $t=1$ second,

$$P=M(v_2-v_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

It should be kept clearly in mind that P is the force *acting upon the mass* whose velocity is changed from v_1 to v_2. Newton's third law of motion states that the mass *reacts* with a force equal in magnitude but opposite in direction to the force P. To avoid confusion the reacting force that water *exerts upon an object* will be designated by F. Hence

$$F=-P=-M(v_2-v_1)=M(v_1-v_2). \quad . \quad . \quad . \qquad (8)$$

Since force and velocity are vector quantities, it follows that if a jet of water impinges against a vane which is either moving or at rest and thereby has its velocity in any direction changed, a force F is exerted upon the vane whose magnitude in any direction is equal to the change in momentum per second that the jet undergoes in the same direction. In other words the force F is equal to the mass impinging per second times the change in velocity in the direction of the force. The X- and Y-components of the force exerted by a jet whose path lies in the XY plane will therefore be,

F_x = Mass impinging per second × change in velocity along the X-axis.

F_y = Mass impinging per second × change in velocity along the Y-axis.

$$F=\sqrt{F_x^2+F_y^2},$$

and the tangent of the angle which this resultant makes with the X-axis is $\dfrac{F_y}{F_x}$.

The change in velocity may be either positive or negative, the only difference being that in the case of a decrease in velocity the dynamic force exerted by the water on the vane is in the same direction as flow, whereas in the case of an increase in the velocity the dynamic force exerted on the vane is opposed to the direction of flow. For instance, referring to Fig. 127, the flow being from the left and the X-component of the velocity being decreased, F_x is directed toward the right, whereas the Y-component of the velocity being increased and the flow being directed upward, F_y is directed downward.

132. Relative and Absolute Velocities.—Strictly speaking, all motion is relative. No object in the universe is known to be fixed in space. An airplane is said to be flying one hundred miles per hour but this is its velocity only with respect to the surface of the earth beneath it. The earth's surface itself is moving at a tremendous speed both with respect to its axis and to the sun, each of which are whirling through space at a still greater rate.

It is nevertheless convenient in connection with this subject to consider all motion with respect to the earth's surface as *absolute* motion. The airplane above referred to has therefore an absolute velocity of 100 miles per hour. Another plane in pursuit may have an *absolute* velocity of 120 miles per hour but its relative velocity with respect to the first plane is only 20 miles per hour. If the two planes were to fly in opposite directions, each retaining its same absolute velocity, the relative velocity between them would be 220 miles per hour. If they were to fly at right angles to each other their relative velocities would be $\sqrt{100^2+120^2} = 156.2$ miles per hour.

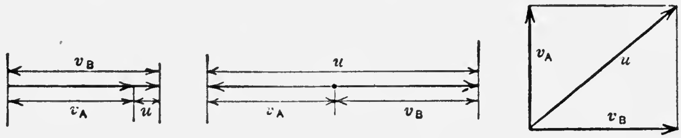

Fig. 125.

Since velocities are vector quantities, these results may be obtained graphically as in Fig. 125, in which,

- v_A is the absolute velocity of A,
- v_B is the absolute velocity of B, and
- u is the relative velocity of either with respect to the other.

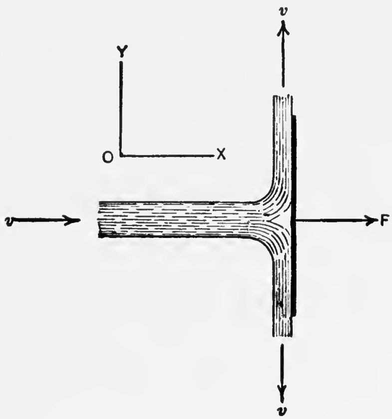

Fig. 126.—Jet impinging against a flat plate.

133. Jet Impinging Normally on a Fixed Flat Plate.—Fig. 126 shows a jet impinging at right angles against a fixed flat

plate. It is assumed that the plate is large enough with respect to the size of the jet so that the jet is deflected through a full 90°. The pressure exerted on the plate varies from a maximum at the axis of the jet where it is very nearly equal to that due to the full velocity head, to zero at a distance approximately equal to the diameter of the jet from its axis. The total pressure exerted is equal to the product of the mass impinging per second and the original velocity of the jet, since the final velocity of the water as it leaves the plate has no component in its original direction. Hence, the force exerted on the plate is,

$$F = Mv = \frac{wQ}{g} v = \frac{wav^2}{g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where M and Q represent respectively the mass and quantity striking the plate per second, and a and v are the cross-sectional area and mean velocity of the jet, all terms being expressed in the foot-pound-second system.

134. Jet Impinging Normally on a Moving Flat Plate.—Consider the case of a jet impinging normally against a flat plate moving in the same direction as the jet or at least having a component of its motion in that direction. It is assumed that the plate has a uniform velocity, being restrained from accelerating by some external agency. The mass impinging per second is

$$M' = \frac{wQ'}{g} = \frac{wau}{g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where Q' is the quantity of water striking the plate in cubic feet per second, a is the cross-sectional area of the jet in square feet and u is the relative velocity of the jet with respect to the plate. The change in velocity is $v - v' = u$, since the velocity in the direction of the force is changed from v to v' and the jet leaves the plate tangentially with a velocity whose X-component is equal to the velocity of the moving plate. The force acting on the plate is

$$F = M'u = \frac{wQ'u}{g} = \frac{wau^2}{g}. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

135. Jet Deflected by a Fixed Curved Vane.—The jet, shown in Fig. 127, is deflected through an angle θ by a fixed, curved, trough-shaped vane AB. It is assumed that the vane is so smooth that friction may be neglected so that the velocity with which the

jet leaves at B may be considered the same as that with which it strikes at A. It is also assumed that the vertical height of the vane is so small that gravity will not appreciably retard the velocity of the jet. Considering the horizontal and vertical components of the force acting on the vane,

$$F_x = M(v - v_x) = \frac{wav}{g}(v - v\cos\theta) = \frac{wav^2}{g}(1 - \cos\theta), \quad . \quad (12)$$

$$F_y = M(0 - v_y) = -\frac{wav}{g}(v\sin\theta) = -\frac{wav^2}{g}\sin\theta. \quad . \quad . \quad (13)$$

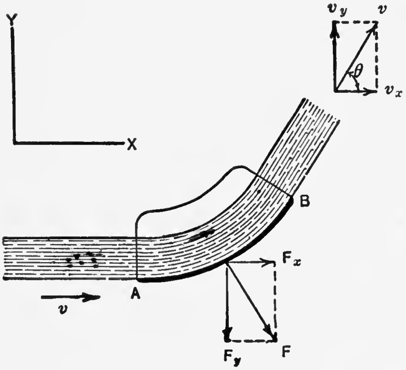

Fig. 127.—Jet impinging on a fixed curved vane.

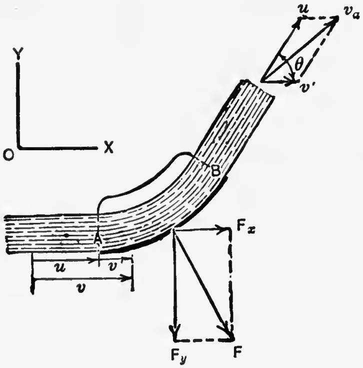

Fig. 128.—Jet impinging on a curved, moving vane.

The negative sign in equation (13) means that the force, F_y, is exerted in a direction opposite to v_y, or in other words, downward.

The resultant force is

$$F = \sqrt{F_x^2 + F_y^2} = \frac{wav^2}{g}\sqrt{2(1 - \cos\theta)} = \frac{2wav^2}{g}\sin\tfrac{1}{2}\theta \quad . \quad (14)$$

If the angle of deflection is greater than 90°, $\cos\theta$, in equation (12) becomes negative. If the jet is deflected through a full 180°, $\theta = -1$, and $\sin\theta = 0$, and the equations become

$$F_x = 2Mv = \frac{2wav^2}{g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$F_y = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

136. Jet Deflected by a Moving Curved Vane.—Consider the vane shown in Fig. 128 to be moving with a uniform velocity v'

in the original direction of the jet. The absolute velocity of the jet as it impinges at A is v and its relative velocity with respect to the vane is $v-v'=u$. The mass impinging per second is therefore $\frac{wau}{g}$. Neglecting friction the relative velocity of the jet with respect to the vane remains unchanged while flowing from A to B so that the jet leaves the vane at B with a relative velocity u in a tangential direction, the X-component of which is $u \cos \theta$. The change in velocity along the X-axis is therefore $u-u \cos \theta$ or $u(1-\cos \theta)$ and

$$F_x=\frac{wau^2}{g}(1-\cos \theta). \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

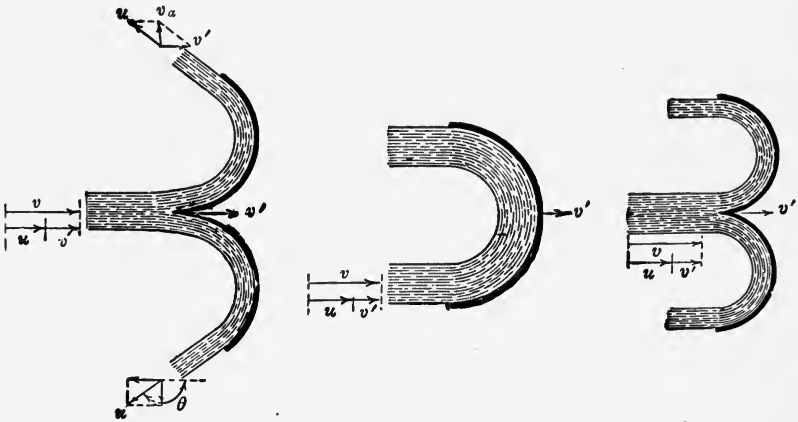

Fig. 129.

In a similar manner the change in velocity in a vertical direction is from zero to $u \sin \theta$ and therefore

$$F_y=-\frac{wau^2}{g} \sin \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The absolute velocity and direction of the jet as it leaves the vane are shown in Fig. 128 by the vector v_a which is the resultant of the relative velocity u and the velocity, v', of the vane.

If a jet is directed against a double-cusped vane as shown in Fig. 129 (a) so that half the jet is deflected by each cusp through equal angles, F_x will be determined by equation (17) but F_y is zero

since the two Y-components balance being equal and opposite in direction.

If a jet is deflected through a full 180° either by a single- or double-cusped vane as shown in Fig. 129 (*b*) and (*c*), obviously, for a stationary vane,

$$F_x = F = 2Mv = \frac{2wav^2}{g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

and for a moving vane

$$F_x = F = 2M'u = \frac{2wau^2}{g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

If a series of vanes are so arranged on the periphery of a wheel that the entire jet, directed tangentially to the circumference is striking either one vane or another successively, the mass impinging again becomes $M = \frac{wav}{g}$ and the force exerted is,

$$F_x = \frac{wav}{g} u(1 - \cos\theta), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

$$F_y = \frac{wav}{g} u \sin\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

It should be noted that when F_y is radial, F_x is the only component of the force tending to produce rotation.

137. Work Done on Moving Vanes.—Since work is equal to force times distance it is apparent that for a jet to do any work upon a vane, the vane must be moving with a velocity between zero and the velocity of the jet since at these limiting velocities either the distance or the force is equal to zero. The question then arises as to what velocity the vane should have, for any given velocity of jet, to perform the maximum amount of work.

The amount of work done per second is the product of the force acting in the direction of motion and the distance through which it acts. Assuming that the direction of motion of the vane is parallel with the direction of the jet, the force acting is (Art. 136),

$$F_x = \frac{wa(v - v')^2}{g}(1 - \cos\theta), \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

and the distance through which it acts per second is equal to the

velocity of the vane, v'. Representing the work done per second, expressed in foot-pounds, by G,

$$G=\frac{wa(v-v')^2}{g}(1-\cos\theta)v'. \quad . \quad . \quad . \quad . \quad (24)$$

Considering v' as the only variable in this expression and equating the first derivative to zero, the relation between v and v' may be determined for which G is a maximum.

$$\frac{dG}{dv'}=\frac{wa(1-\cos\theta)}{g}(v^2-4vv'+3v'^2)=0,$$

from which

$$v'=v \quad \text{and} \quad v'=\frac{v}{3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

When $v'=v$, no work is done since the force exerted is then zero and this value represents a condition of minimum work. For maximum work therefore $v'=\frac{v}{3}$.

In the case of a series of vanes so arranged that the entire jet strikes either one vane or another successively, the force exerted in the direction of motion, which is assumed parallel with the direction of the jet is (Art. 136),

$$F_x=\frac{wav}{g}(v-v')(1-\cos\theta). \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

The distance through which this force acts in one second is v', and therefore,

$$G=\frac{wav}{g}(v-v')(1-\cos\theta)v'. \quad . \quad . \quad . \quad . \quad (27)$$

Differentiating, and equating to zero,

$$\frac{dG}{dv'}=\frac{wav(1-\cos\theta)}{g}(v-2v')=0,$$

and for maximum work

$$v'=\frac{v}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Substituting this value of v' in equation (27),

$$G=\frac{wav^3}{4g}(1-\cos\theta), \quad \text{(29)}$$

or

$$G=\frac{Mv^2}{4}(1-\cos\theta), \quad \text{(30)}$$

which is $\frac{(1-\cos\theta)}{2}$ times the total kinetic energy available in the jet. For $\theta=180°$ this expression equals unity and

$$G=\frac{Mv^2}{2}, \quad \text{(31)}$$

the total kinetic energy of the jet being converted into work. This also appears from considering that the relative velocity of the jet as it leaves the vane is $\frac{v}{2}$, which is also the velocity of the vane. These two velocities being equal and opposite in direction have a resultant of zero. The water thus leaves the vane with zero velocity, signifying that all of its original energy has been utilized in performing work.

The above principles are made use of in the design of impulse turbines, which consist of a series of vanes attached to the periphery of a wheel. The angle θ must be somewhat less than 180° so that the jet in leaving a vane will not interfere with the succeeding vane. Making the angle θ equal to 170° in place of 180° reduces the force applied to the wheel by less than 1 per cent.

138. Forces Exerted upon Pipes.—In the preceding articles of this chapter the discussion has been restricted to forces exerted by jets impinging against flat and curved surfaces. As it was always considered that the flow was free and unconfined the only forces acting were dynamic.

Consideration will now be given to the longitudinal thrust exerted upon a section of pipe by water flowing through it under pressure. This thrust will usually be found to be the resultant of both static and dynamic forces. The transverse forces which determine the necessary thickness of pipe were discussed in Art. 29.

139. Straight Pipe of Varying Diameter.—Under conditions of steady flow through a straight pipe of varying diameter there is a

longitudinal thrust exerted upon the pipe. This thrust is the resultant of a dynamic force, a static pressure, and frictional resistance.

Fig. 130 shows a straight section of converging pipe. Let p_1, a_1, and v_1 represent respectively the pressure, area, and mean velocity at AB and p_2, a_2, and v_2 the corresponding values at CD. In flowing from AB to CD the water is accelerated from v_1 to v_2 and the force, P, producing this acceleration is the resultant of all the component forces acting on the mass $ABCD$. These forces consist of the pressures on the sections AB and CD, the pressure exerted by the pipe walls $ACBD$ and the force of gravity, the last

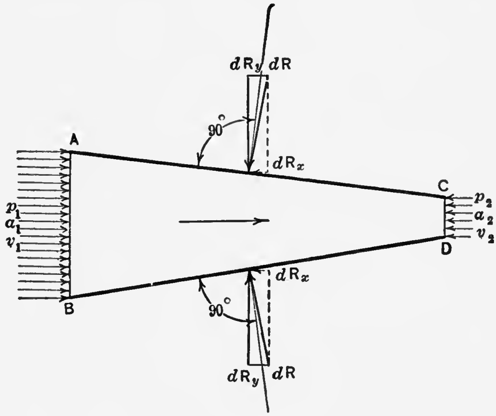

Fig. 130.

of which can be neglected since it acts vertically and has no component in the direction of acceleration. The pressures on AB and CD are a_1,p_1 and a_2p_2 respectively, a_1p_1 acting in the direction of acceleration and a_2p_2 being opposed to it. The pressure, dR, exerted on the water by any differential area, da, of the pipe walls will be inclined slightly from the normal toward the direction of flow on account of friction. The vertical component of dR, being normal to the direction of acceleration may be neglected, leaving dR_x as the only component to be considered. All the values of dR_x for the various elementary areas of pipe wall, being parallel and acting in the same direction, may be combined into the resultant, R_x, whose magnitude is as yet unknown but whose

direction is opposed to acceleration. Therefore the force producing acceleration,

$$P = a_1 p_1 - a_2 p_2 - R_x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

But in Art. 131 it was shown that

$$P = M(v_2 - v_1) = \frac{w a_1 v_1}{g}(v_2 - v_1), \quad . \quad . \quad . \quad . \quad . \quad (33)$$

and therefore, from equations (32) and (33),

$$R_x = a_1 p_1 - a_2 p_2 - \frac{w a_1 v_1}{g}(v_2 - v_1). \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Since R_x is the X-component of the forces exerted upon the water by the pipe it follows that the thrust exerted upon the section of pipe by the water must be equal and opposite to R_x, or in other words the thrust will act toward the right in Fig. 130.

Considering a straight section of pipe of constant diameter throughout, equation (34) reduces to

$$R_x = a(p_1 - p_2), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

since $a_1 = a_2 = a$, and $v_1 = v_2$. In equation (35) $p_1 - p_2$ is the drop in pressure resulting from friction between sections AB and CD.

140. Pipe Bends.—The thrust exerted upon a curved section of pipe of either constant or varying diameter is the resultant of component forces similar to those discussed in the preceding article. The chief difference lies in the fact that the resultant thrust on a curved section of pipe has both X- and Y-components since there is a change in velocity along both of these axes.

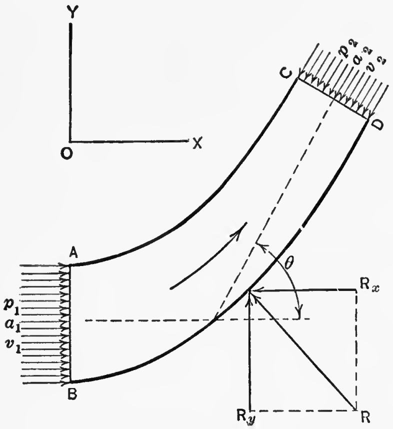

FIG. 131.

Fig. 131 shows a pipe bend, having a diameter decreasing from AB to CD and a deflection angle, θ. Let R_x and R_y represent the X- and Y-components of the forces exerted by the pipe upon the water.

The X- and Y-components of the thrust exerted by the water upon the pipe are equal in magnitude but opposite in direction to R_x and R_y respectively. Assuming that the bend lies in a horizontal plane so that the action of gravity is normal to the direction of acceleration and therefore may be ignored, the resultant X and Y forces producing acceleration are,

$$P_x = a_1p_1 - a_2p_2 \cos\theta - R_x = \frac{wQ}{g}(v_2 \cos\theta - v_1), \quad . \quad (36)$$

$$P_y = -a_2p_2 \sin\theta + R_y = \frac{wQ}{g} v_2 \sin\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

the right-hand members in these equations representing the increase in momentum along the X- and Y-axes resulting from the accelerating forces.

From the above equations,

$$R_x = a_1p_1 - a_2p_2 \cos\theta + \frac{wQ}{g}(v_1 - v_2 \cos\theta), \quad . \quad . \quad . \quad (38)$$

$$R_y = a_2p_2 \sin\theta + \frac{wQ}{g} v_2 \sin\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

If the pipe bend is one of constant diameter throughout, $a_1 = a_2$, $v_1 = v_2$, and $p_1 = p_2$ (approximately), and the equations reduce to

$$R_x = \left(ap + \frac{wav^2}{g}\right)(1 - \cos\theta), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

$$R_y = \left(ap + \frac{wav^2}{g}\right) \sin\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

If the angle θ equals 90° these equations become

$$R_x = R_y = ap + \frac{wav^2}{g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

141. Water Hammer in Pipe Lines.—In Fig. 132 is shown a pipe line leading from a reservoir, A, and discharging into the air at B near which is located a gate valve. If the valve is suddenly closed a dynamic pressure is at once exerted in the pipe in excess of the normal static pressure. The magnitude of this pressure is frequently much greater than that of any static pressure to which the pipe may ever be subjected and the possibility of the occurrence

of such pressure must therefore be investigated in connection with the design of any pipe line of importance.

This dynamic pressure, commonly called *water hammer*, is the result of a sudden transformation of the kinetic energy of the moving mass of water within the pipe into pressure energy. Since force equals mass times acceleration, or

$$P=M\frac{dv}{dt}, \qquad \ldots \ldots \ldots \qquad (43)$$

it follows that if the velocity of the mass M could be reduced from v to zero instantaneously, this equation would become

$$P=M\frac{v}{0}, \qquad \ldots \ldots \ldots \qquad (44)$$

or in other words the pressure resulting from the change would be infinite. Such an instantaneous change is, however, impossible.

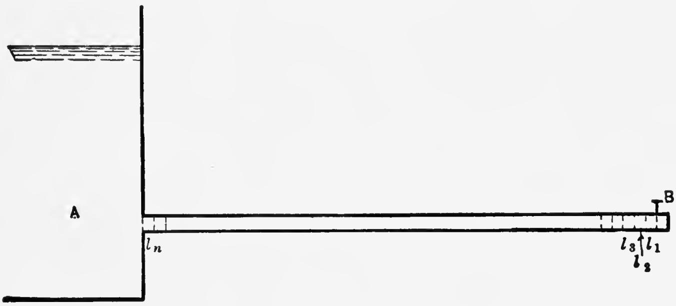

Fig. 132.

Consider the conditions within the pipe immediately following the closure of the valve. Let $l_1, l_2, l_3, \ldots l_n$ represent infinitesimally short sections of pipe as shown in Fig. 132. The instant the valve is closed, the water in contact with it in section l_1 is brought to rest, its kinetic energy is transformed into pressure energy, the water is somewhat compressed and the pipe wall with which it is in contact expands slightly as a result of the increased stress to which it is subjected. Because of the enlarged cross-sectional area of l_1 and the compressed condition of the water within it, a greater mass of water is now contained within this section than before the closure. It is evident then that a small volume of water flowed into section l_1 after the valve was closed. An

instant later a similar procedure takes place in l_2 and then in l_3, so that evidently a wave of increased pressure travels up the pipe to the reservoir. The instant this wave reaches the reservoir the entire pipe is expanded and the water within it is compressed by a pressure greater than that due to the normal static head. There is now no longer any moving mass of water within the pipe, the conversion of whose kinetic energy into pressure energy serves to maintain this high pressure and therefore the pipe begins to contract and the water to expand with a consequent return to normal static pressure. This process starts at the reservoir and travels as a wave to the lower end. During this second period some of the water stored within the pipe flows back into the reservoir but on account of the inertia of this moving mass an amount flows back greater than the excess amount stored at the end of the first period so that the instant this second wave reaches the valve the pressure at that point drops not only to the normal static pressure but below it. A third period now follows during which a wave of pressure less than static sweeps up the pipe to the reservoir. When it reaches the reservoir the entire pipe is under pressure less than static but since all the water is again at rest the pressure in l_n immediately returns to the normal static pressure due to the head of water in the reservoir. This starts a fourth period marked by a wave of normal static pressure moving down the pipe. When the valve is reached the pressure there is normal and for an instant the conditions throughout the pipe are similar to what they were when the valve was first closed. The velocity of the water (and the resultant water hammer) is now, however, somewhat less than it was at the time of closure because of friction and the imperfect elasticity of the pipe and the water.

Instantly another cycle begins similar to the one above described, and then another, and so on, each set of waves successively diminishing, until finally the waves die out from the influences above mentioned.

Equation (44) shows that for instantaneous closure of valve the pressure created would be infinite if the water were incompressible and the pipe were inelastic. Instantaneous closure is, however, physically impossible. To determine the amount of excess pressure actually resulting from water hammer it is necessary to take into consideration the elasticity of the pipe and the compressibility of water. This leads to a rather lengthy mathe-

matical analysis which will here be avoided and there will be given only the resulting workable equations. The following nomenclature will be used, all units being expressed in feet and seconds except E and E', which are in pounds per square inch:

b = thickness of pipe walls;
D = inside diameter of pipe;
E = modulus of elasticity of pipe walls in tension;
E' = modulus of elasticity of water in compression;
g = acceleration of gravity;
h = head due to water hammer (in excess of static head);
H = normal static head in pipe;
L = length of pipe line;
T = time of closing valve;
v = mean velocity of water in pipe before closure of valve;
v_w = velocity of pressure wave along pipe.

142. Formulas for Water Hammer.—In the following discussion whenever the term " pressure " is used it is understood to mean " pressure due to water hammer " and is the amount of pressure in excess of that due to the normal static head.

If the valve is closed instantaneously the pressure in l_1 immediately rises to p_{max} and remains at this value while the pressure wave travels to the reservoir and returns. The time required for the wave to travel to the reservoir and back to the valve is $\frac{2L}{v_w}$. The pressure in l_n reaches this same p_{max} but remains at that value only for an instant. At any intermediate section, p_{max} is maintained only until the wave of reduced pressure reaches that section. If the time taken to completely close the valve is exactly $\frac{2L}{v_w}$, p_{max} will occur only in section l_1, and will last only for an instant, being immediately lowered by the return of the static pressure wave. If the time of closing the valve is greater than $\frac{2L}{v_w}$, p_{max} will never be attained since the wave of reduced pressure will then have reached l_1 before the valve is completely closed or p_{max} is reached.

Evidently two formulas are necessary; one to determine the maximum water hammer, when the time of closure is less than

$\frac{2L}{v_w}$, and the other to determine the ordinary water hammer that occurs when the time of closure is greater than $\frac{2L}{v_w}$, as is usually the case.

The same formula for the determination of maximum water hammer has been quite generally adopted. The general expression is,

$$h_{\max}=\frac{vv_w}{g}, \qquad (45)$$

where

$$v_w=\frac{4660}{\sqrt{1+\frac{E'D}{Eb}}}. \qquad (46)$$

Substituting this value in equation (45),

$$h_{\max}=\frac{145v}{\sqrt{1+\frac{E'D}{Eb}}}. \qquad (47)$$

For steel pipe this reduces to

$$h_{\max}=\frac{145v}{\sqrt{1+0.01\frac{D}{b}}}. \qquad (48)$$

Frequently these expressions appear in different forms but they may all be reduced to the above forms.

There is no such general agreement as to the proper formula to be used for the determination of ordinary water hammer when T is greater than $\frac{2L}{v_w}$. Many formulas have been derived, some giving results twice as great as others. Certain assumptions as to the manner of valve closure, the effect of friction and the manner in which the waves are reflected, etc., must be made before any theoretical formula can be derived. It appears that the main reason for the wide discrepancy in results lies in the difference in these fundamental assumptions.

Assuming that the valve is closed in such manner that the rate of rise in pressure will be constant throughout the entire closure, OA, Fig. 133, represents the variation in pressure at the

valve during the time T, provided that no return pressure wave interferes. After a time, $\frac{2L}{v_w}$, however, the returning wave will reach the valve and assuming that its intensity has been undiminished by friction or other cause, it will exactly annul the tendency for the pressure to increase, due to continued closing, and as a result the pressure remains constant during the remainder

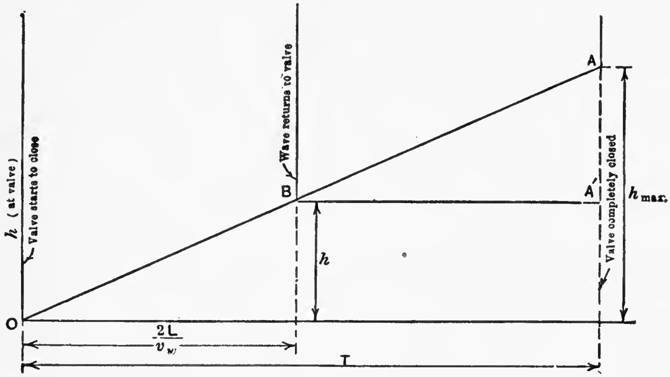

FIG. 133.

of the time T, as shown by the horizontal line BA'. From similar triangles,

$$h : \frac{vv_w}{g} = \frac{2L}{v_w} : T.$$

From which

$$h = \frac{2Lv}{gT}. \quad \ldots \ldots \ldots \ldots \quad (49)$$

This formula was first proposed by Professor Joukovsky [1] of Moscow, Russia, in 1898 and, it is claimed, was substantiated by a series of experiments which he conducted.

Other commonly used formulas [2] are as follows:

$$h = \frac{NH}{2} + H\sqrt{\frac{N^2}{4} + N}, \quad \ldots \ldots \quad \text{(Allievi)}$$

[1] N. JOUKOVSKY: Trans. of Prof. N. Joukovsky's paper on Water Hammer, by Boris Simin; Journal American Waterworks Association (1904).

[2] MILTON M. WARREN: Penstock and Surge-Tank Problems. *Trans. Amer. Soc. Civ. Eng.*, vol. 79 (1915). Contains discussions of all commonly used water hammer formulas.

where

$$N=\left(\frac{Lv}{gHT}\right)^2,$$

$$h=\frac{2MH}{N^2}(M+\sqrt{M^2+N^2}), \quad . \quad . \quad . \quad \text{(Johnson)}$$

where

$$M=Lv \quad \text{and} \quad N=2gHT,$$

$$h=\frac{Lv}{g\left(T-\frac{L}{v_w}\right)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(Warren)}$$

$$h=\frac{Lv}{gT}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(Mead, et al.)}$$

Joukovsky's formula gives results twice as great as Mead's formula. Johnson's formula can readily be reduced to Allievi's. It may be noted that these two formulas make h vary with H, which the other formulas do not do. Justification for this variation is not apparent.

Eliminating Johnson's formula (which reduces to Allievi's), no two of the above formulas give results that are at all similar under all conditions. Discrepancies of from 100 to 200 per cent are possible. A comprehensive and carefully conducted series of experiments are necessary before any formula for ordinary water hammer can be relied upon to give trustworthy results.

PROBLEMS

1. A jet 1 in. in diameter and having a velocity of 25 ft. per second strikes normally against a fixed, flat plate. Determine the pressure on the plate.

2. In Problem 1, what would be the pressure on the plate if it were moving with a uniform velocity of 10 ft. per second in the same direction as the jet?

3. What should be the velocity of the plate, in Problem 2, if the jet is to perform the maximum amount of work? Determine the corresponding amount of work in foot-pounds per second.

4. A jet having a diameter of 2 in. and a velocity of 40 ft. per second is deflected through an angle of 60° by a fixed, curved vane. Determine the X- and Y-components of the force exerted.

5. Solve Problem 4 if the vane is moving with a velocity of 25 ft. per second in the same direction as the jet.

6. A $1\frac{1}{2}$-in. nozzle has a coefficient of velocity of 0.97 and a coefficient of contraction of unity. The base of the nozzle has a diameter of 4 in., at which point the gage pressure is 80 lbs. per square inch. The jet is

deflected through an angle of 150° by a double-cusped vane that has a velocity, in the direction of the jet of 30 ft. per second. What is the pressure exerted on the vane and what is the amount of work done expressed in foot-pounds?

7. In Problem 6 determine the velocity the vane must have if the jet is to perform the maximum amount of work. What is the maximum work in foot-pounds?

8. If the jet, in Problem 6, strikes a series of vanes so arranged on the periphery of a wheel that the entire jet is deflected through an angle of 170°, what is the maximum amount of work that can be done?

9. A horizontal straight pipe gradually reduces in diameter from 12 in. to 6 in. If, at the larger end, the gage pressure is 40 lbs. per square inch and the velocity is 10 ft. per second, what is the total longitudinal thrust exerted on the pipe? Neglect friction.

10. A bend in a pipe line gradually reduces from 24 in. to 12 in. The deflection angle is 60°. If at the larger end the gage pressure is 25 lbs. per square inch and the velocity is 8 ft. per second, determine the X- and Y-components of the dynamic thrust exerted on the bend. Also determine the X- and Y-components of the total thrust exerted on the bend, neglecting friction.

11. A 24-in. cast-iron pipe $\frac{3}{4}$ in. thick and 6000 ft. long discharges water from a reservoir under a head of 80 ft. What is the pressure due to water hammer resulting from the instantaneous closure of a valve at the discharge end?

12. If the time of closing the valve, in Problem 11, is 6 sec., determine the resulting pressure due to water hammer, comparing results obtained by use of formulas by Joukovsky, Johnson, Warren and Mead.

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